



MADURAI KAMARAJ UNIVERSITY



(University with Potential for Excellence)

DISTANCE EDUCATION

**B.Sc. SECOND YEAR
MATHEMATICS**

**PART III
PAPER - III**

**MODERN ALGEBRA
AND
DIFFERENTIAL EQUATIONS**

Recognised by D.E.C.

www.mkudde.org

R 84

ACC-MKU
01178

SYLLABUS

MODERN ALGEBRA AND DIFFERENTIAL EQUATIONS

MODERN ALGEBRA

Unit - 1 Relations - Definitions - Types of relations - Functions - Types of Functions, Binary Operations - Peano's Postulates - Principles of Induction - Simple Problems - Law of Trichotomy.

Unit - 2 Subgroups - definitions and examples - center - Normalizer - Intersection and Union of Subgroups - Permutations - Cycles and Transpositions - Permutations as a product of disjoint cycles and Transpositions - even and odd Permutations - S_n and A_n - Cyclic groups - definitions and examples - cyclic groups and abelian - A group is cyclic if its order is equal to the order of one its elements - subgroups of cyclic groups are cyclic - Theorem on the number of generators of cyclic groups.

Unit - 3 Cosets and their properties - Congruence relation module a subgroup - Lagrange's theorem and its consequences - The order of a element of a finite group divides the order of the group - A group of prime order is cyclic - a group has no proper subgroup if it is a cyclic group of prime order - Euler's theorem - Fermat's theorem.

Normal Subgroups - equivalent conditions for a subgroup to be normal - any subgroup of an abelian group is normal - a subgroup of index 2 is normal intersection of two normal subgroups - intersection of a subgroup and a normal subgroup - center is a normal subgroup - If a subgroup has exactly one subgroup of given then it is normal - Quotient group.

Unit - 4 Homomorphism - type of homomorphism with reference to identity, inverse and order of an element - its properties - Kernel of a homomorphism - homomorphic image of an abelian group is abelian and that of a cyclic group is cyclic group is cyclic - Isomorphism - Isomorphism is an equivalence relation among groups - any finite cyclic group of order n is isomorphic to Z_n - Cayley's theorem. The fundamental theorem of homomorphism.

Unit - 5 Rings - definitions and examples - elementary properties of rings - Division rings and fields - Zero divisors of Z_n . Integral domain - Cancellation laws- a field is an integral domain - a finite integral domain is a field - characteristic of a ring - characteristic of integral domain is either Zero or a prime number - Field of quotients of an integral domain - every integral domain can be embedded in a field.

DIFFERENTIAL EQUATIONS

Unit - 1 Equations of the first order, but of higher degree - Equations solvable for y solvable by x , solvable for p -Clairant's form - Equations that do not contain x , y explicitly - Equations homogeneous in x and t - Linear equation with constant coefficients.

Unit - 2 Linear equations with variable coefficients - Equations reducible to the linear homogeneous equations - simultaneous linear differential equations.

Unit - 3 Linear equations of the second order - Reduction of the normal form of Removing the first derivative method - variation of parameters - Total differential equation - Rules for integrating $Pdx + Qdy + Rdz = 0$.

Unit - 4 Partial differential equations of the first order - classification of integrals - Derivations of Partial differential equation - Lagrange's methods of solving - the linear equations - standard form - Equation reducible to the standard forms.

Unit - 5 Laplace Transform - Theorems - Problems - Evaluation of integrals. Inverse Laplace transforms - Results - Problems - Solving ordinary differential equation with constant coefficients and variable coefficients and simultaneous linear equations using Laplace Transform.

Books for Reference

- 1) Modern Algebra by S. Arumugam & Others
- 2) Differential Equations by S. Narayanan, T.K. Manickavasagam Pillai

SCHEME OF LESSONS

MODERN ALGEBRA

UNIT - 1

5-36

- 1) Relations
- 2) Functions
- 3) Binary Operations
- 4) Peano's Postulates
- 5) Law of Trichotomy
- 6) Mathematical Induction

UNIT - 2

37-65

- 1) Subgroups
- 2) Permutations
- 3) Cyclic groups

UNIT - 3

66-81

- 1) Cosets and Lagrange's theorem
- 2) Euler Theorem and Fermat's Theorem
- 3) Normal subgroups and quotient groups

UNIT - 4

82-93

- 1) Homomorphism of groups
- 2) Fundamental theorem on homomorphisms
- 3) Cayley's theorem

UNIT - 5

94-104

- 1) Ring - Definition and Examples
- 2) Integral Domains, Fields and Division Rings

DIFFERENTIAL EQUATIONS

UNIT - 6

105-136

- 1) Equation of First order
- 2) Equations solvable for p, x or y .
- 3) Clairant's form
- 4) Equations that do not contain x or y .
- 5) Linear Equation with constant coefficients.

UNIT - 7

137-148

- 1) Linear Equation with variable coefficients
- 2) Equation reducible to the linear homogenous Equation
- 3) Simultaneous linear Differential Equations.

UNIT - 8

149-160

- 1) Linear Equations of the second order
- 2) Reduction to the normal form
- 3) Variation of parameters
- 4) Total differential Equation
- 5) Rules for integrating $pdx + Qdy + Rdz = 0$.

UNIT - 9

161-177

- 1) Partial Differential Equations
- 2) Classification of Integrals
- 3) Standard Forms

UNIT - 10

178-202

- 1) Laplace Transforms
- 2) Inverse Laplace Transforms
- 3) Solving ordinary Differential Equations using Laplace Transforms

MODERN ALGEBRA

UNIT - I

1.1. Relations:

A relation is concerned with the existence or non existence of some type of bond between certain ordered pairs. In other words a relations provides a criterion for distinguishing certain ordered pairs from others. For example consider the relation "less than" in the set of N of natural numbers. The ordered pair $(2,3)$ is such that 2 is in the given relation to 3 (ie) $2 < 3$. Whereas the ordered pair $(3,2)$ is not in the given relation. This we can collect all ordered pairs in $M \times N$ which are in the given relation. Clearly this subset of $N \times N$ is given by $\rho = \{(x,y) / x,y \in N \text{ and } x < y\}$ and this subset of new specifies the relation completely in the sense that $\{(x,y) \in \rho \text{ iff } x < y$.

Definition:

Let A and B be non empty sets. A subset ρ of $A \times B$ is called a relation (or) a binary relation from A to B . Of an ordered pair $(a,b) \in \rho$ we say that a is related to b in the given relation and we write $a \sim b$. If $(a,b) \notin \rho$. We say that a is not related to b in the given relation.

Though we have defined a relation strictly in set theoretical terms a relation on a set can be specified by a Kerbal phrase (or) a mathematical symbol in such a way that given an ordered pair (a,b) it must be possible for us to decide whether 'a' new the given relation to b (or) not. In what follows we confine ourselves to relations defined from a set 's' itself.

Examples:

1) $S = \{1,2,3,4\}$ $\rho = \{(1,2) (2,3) (3,3) (1,4)\}$ is relation on S .

2) $S = Z$: $a \rho b$ means $a < b$.

3) $S = Z$ $a \rho b$ means $a \leq b$

4) $S = Z$ $a \rho b$ means a divides b

5) $S = Z$ $a \rho b$ means $a \equiv b \pmod{m}$ (ie) $a - b$ is a multiple of m .

6) $S = Z$ $a \rho b$ means ab is even.

7) $S = Z$ $a \rho b$ means ab is odd.

8) $S = Z$ $a \rho b$ means ab is a perfect square.

9) $S = P(A)$ $B \rho C$ means $B \subseteq C$.

10) $S = R$ $a \rho b$ means $a = b$

11) $S = R$ $a \rho b$ means $a^3 = b$.

12) $S = \mathbb{R}$ aRb means $a-b$ is an integer

13) $S =$ The set of all lines in the Euclidean plane $R \times R$ aRb means a is parallel to b .

14) $S =$ The set of all lines in the Euclidean plane $R \times R$ aRb means a is perpendicular to b .

15) $S =$ The set of all students who appeared for the B.Sc., Mathematics examination of the Madurai Kamaraj University in April 1985 aRb means a and b obtain equal marks in the Modern Algebra paper.

16) $S = \mathbb{R}$ aRb means $|a| = |b|$

17) $S = \mathbb{C}$ aRb means $|a| = |b|$

Equivalence Relations:

Definition :

A relation r defined on a set S is said to be symmetric if aRb means $\Rightarrow bRa$.

Examples:

1) Consider the example

$S =$ The set of all lines in the Euclidean plane $R \times R$. Let aRb . Then a is perpendicular to b and hence b is perpendicular to a . Hence bRa thus $aRb \rightarrow bRa$. Hence r is symmetric.

Definition:

A relation r defined on a set S is said to be reflexive if aRa for all $a \in S$.

Examples:

1. Consider the examples

$S = \mathbb{Z}$ aRb means ab is odd. Here aRa is not true for any even integer. Hence r is not reflexive.

2. $S =$ The set of all lines in the Euclidean plane $R \times R$.

aRb means a is parallel to b . Clearly any line is parallel to itself. Thus aRa for all $a \in S$.

Hence r is reflexive.

Definition:

A relation r defined on a set S is said to be transitive if aRb and bRc .

$\Rightarrow aRc$.

Examples:

Consider the example.

$S = \mathbb{Z}$ arb means $a \leq b$.

Let arb and brc

$\setminus a \leq b$ and $b \leq c$

$\setminus a \leq c$ and hence arc

$\setminus r$ is transitive.

Definition :

A relation r defined on a set S is said to be an equivalence relation if r is reflexive symmetric and transitive.

Examples :

$S = \mathbb{Z}$; arb means $a \equiv b \pmod{m}$.

(b) $a-b$ is a multiple of m .

For any $a \in \mathbb{Z}$: $a - a = 0$ which is a multiple of m . Hence ara.

$\setminus r$ is reflexive.

set arb then $a-b = km$ where $K \in \mathbb{Z}$

$\setminus b-a = (-k)m$ and hence bra.

$\setminus r$ is symmetric.

Let arb and brc.

$a-b = km$ and $b-c = lm$ where $k, l \in \mathbb{Z}$.

$\setminus a-c = (k+l)m$, and hence arc

$\setminus r$ is transitive

Thus r is an equivalence relation.

Definition:

Let r be an equivalence relation defined on a set s . Let $x \in s$. The equivalence class $\{x\}$ determined by the element x is defined by $[x] = \{y \in s / x r y\}$

Since $x r x$, $x \in [x]$, so that any equivalence class is non-empty.

Examples :

1) Consider the relation r defined on \mathbb{Z} by

$x\rho y \Rightarrow x-y$ is a multiple of 3.

$$[0] = \{y \in \mathbb{Z} / y-0 = 3k \text{ where } k \in \mathbb{Z}\}$$

$$= \{0, \pm 3, \pm 6, \dots, \pm 3k, \dots\}$$

$$\text{Similarly } [1] = \{3K+1 / K \in \mathbb{Z}\} = \{-5, -2, 1, 4, 7, \dots\}$$

$$[2] = \{3K+2 / K \in \mathbb{Z}\} = \{\dots, -4, -1, 2, 5, 8, \dots\}$$

$$[3] = \{3K+3 / K \in \mathbb{Z}\} = [0] = [6] = [9] = \dots$$

In fact it is easy to see that $[0], [1], [2]$ are the only three distinct equivalence classes. Any two distinct equivalence classes are disjoint and the union of all these equivalence classes is equal to \mathbb{Z} .

Definition :

Let ρ be an equivalence relation defined on a set S . The set of all equivalence classes is called the quotient set of S and is denoted by S/ρ .

Definition :

Let S be any set. A collection of pairwise disjoint non empty subset of S whose union is S is called a partition of S .

Examples:

Let $S = \{1, 2, 3, 4, 5\}$. The subsets $\{1\} \{2\} \{3, 4\} \{5\}$ form a partition of S .

Theorem :

Let ρ be an equivalence relation defined on a set S then.

(i) $a\rho b \Leftrightarrow [a] = [b]$

(ii) Any two distinct equivalence classes are disjoint.

(iii) S is the union of all the equivalence classes.

(ie) The set of all equivalence classes forms a partition of S .

Pf:

(i) let $a\rho b$ we shall prove that

$$[a] = [b]$$

Let $x \in [a]$ then $x\rho a$.

Since $a\rho b$ by transitivity we get $x\rho b$

Hence $x \in [b]$ so that $[a] \subseteq [b]$

Similarly $[b] \subseteq [a]$ Hence $[b] = [a]$

conversely Let $[a] = [b]$.

Then a and b belong to the same equivalence class.

Hence $a \rho b$.

(ii) It is enough if we prove that

$$[a] \cap [b] \neq \emptyset \Rightarrow [a] = [b]$$

Let $[a] \cap [b] \neq \emptyset$

Let $C \in [a] \cap [b]$

Then $C \in [a]$ and $C \in [b]$

$\therefore C \rho a$ and $C \rho b$

$\therefore a \rho c$ and $c \rho b$

$\therefore a \rho b$ and hence $[a] = [b]$

(iii) Since each element a of S is in $[a]$. The union of all equivalence classes is S .

Note :

The above theorem shows that every equivalence relation defined on a set S gives rise to a partition of S . The following theorem deals with the converse situation.

Theorem:

Any partition of a set S determines an equivalence relations ρ such that the members of the partition are precisely the equivalence classes defined by ρ .

Pf :

If $a, b \in S$ we define $a \rho b \Leftrightarrow a$ and b belong to the same member of the partition obviously ρ is reflexive and symmetric.

Now let $a \rho b$ and $b \rho c$

$a \rho b \Leftrightarrow a$ and b belong to the same partition set A .

$b \rho c \Leftrightarrow b$ and c belong to the same partition set B .

Suppose $A \neq B$.

Since $b \in A$ and $b \in B$

$A \cap B \neq \emptyset$ This is a contradiction since any two partition sets are disjoint. Hence $A = B$. Thus a and $c \in A$ and so that $a \rho c$.

Hence ρ is transitive. Thus ρ is an equivalence relation.

Now let $a \in S$.

Let A be the unique member of the partition such that $a \in A$

Then $[a] = A$ (by definition of ρ)

Solved Problems :

1) Find the equivalence relation induced by the partition $\{\{1\}, \{2,3\}, \{4\}\}$ of $S = \{1,2,3,4\}$

Solution:

The equivalence relation ρ induced by the given partition is given by the following subset of $S \times S$ $\{(1,1) (2,2) (3,3), (2,3) (3,2) (4,4)\}$.

2) Find the equivalence relation induced by the partition $\{A, B\}$ of Z where $A = \{0, 1, 2, \dots\}$

$B = \{-1, -2, -3, \dots\}$

Solution :

Let $x, y \in Z$

Then $x \rho y \Leftrightarrow x, y \in A$ (or) $x, y \in B$

$\therefore x \rho y \Leftrightarrow x, y \geq 0$ (or) $x, y < 0$.

3) If ρ and σ are equivalence relations defined on a set S . Prove that $\rho \cap \sigma$ is an equivalence relation.

Pf :

Let $x \in S$

Then $x \rho x$ and $x \sigma x$ (Since ρ and σ are reflexive)

$\therefore x (\rho \cap \sigma) x$ Hence $\rho \cap \sigma$ is reflexive.

Let $x (\rho \cap \sigma) y$. Then $x \rho y$ and $x \sigma y$

$y \rho x$ and $y \sigma x$ (since ρ, σ are symmetric)

$\therefore y (\rho \cap \sigma) x$ Hence $\rho \cap \sigma$ is symmetric

Let $x (\rho \cap \sigma) y$ and $y (\rho \cap \sigma) z$

Then $(x \rho y$ and $x \sigma y)$ and $(y \rho z$ and $y \sigma z)$

$\therefore (x \rho y$ and $y \rho z)$ and $(x \sigma y$ and $y \sigma z)$

$\therefore (x \rho z)$ and $(x \sigma z)$ (Since ρ, σ are transitive)

$\therefore x (\rho \cap \sigma) z$ Hence $\rho \cap \sigma$ is transitive

4) Show that the union of two equivalence relations need not be an equivalence relation.

Pf:

Let $S = \{1, 2, 3\}$

Let $\rho = \{(1, 1) (2, 2) (3, 3) (1, 2) (2, 1)\}$

$\sigma = \{(1, 1) (2, 2) (3, 3) (2, 3) (3, 2)\}$

Clearly ρ and σ are equivalence relations on S .

Now $\rho \cup \sigma = \{(1, 1) (2, 2) (1, 2) (3, 3) (2, 1) (2, 3) (3, 2)\}$

$\therefore \rho \cup \sigma$ is not transitive.

Since $(1, 2) (2, 3) \in \rho \cup \sigma$ but $(1, 3) \notin \rho \cup \sigma$.

$\therefore \rho \cup \sigma$ is not an equivalence relation.

5) What are the smallest and largest equivalence relations on a set S .

Pf:

Any relation on S is a subset of $S \times S$. Consider the subset Δ of $S \times S$ given by $\Delta = \{(x, x) / x \in S\}$ clearly Δ is an equivalence relation on S . Now let ρ be any other equivalence relation on S .

Since ρ is reflexive. ρ contains Δ . Hence Δ is the smallest equivalence relation on S . Obviously the largest equivalence relation on S is given by the subset $S \times S$.

6) Let A be a set with n elements.

(i) Find the number of relations that can be defined on A .

(ii) Find the number of reflexive relations that can be defined on A .

Pf:

(i) Any relation on A is a subset of $A \times A$. Since A has n elements. $A \times A$ has n^2 elements.

\therefore Number of relations that can be defined on A = number of subsets of $A \times A = 2^{n^2}$

(ii) Let $\Delta = \{(a, a) / a \in A\}$. Any reflexive relation on A is of the form $\Delta \cup B$.

Where B is any subset of $(A \times A) - \Delta$

Further $(A \times A) - \Delta$ has $n^2 - n$ elements.

∴ Number of reflexive relations on A

$$= \text{number of subsets of } (A \times A) - \Delta = 2^{n^2 - n}$$

Exercises :-

- 1) Prove that $[a] \cap [b] = \phi \Leftrightarrow a$ is not related to b .
- 2) Prove that $[a] \leq [b] \Leftrightarrow [a] = [b]$

1.2 Functions

The notion of a function is basic in all branches of mathematics. In elementary calculus Y is said to be a function of X if when x is given, y is determined uniquely. Thus $y = x^2$ is a simple example of a function and this function is given by a rule in the form of an algebraic expression. This function is defined for all real numbers. However the functions $y = 1/x$; $y = \log x$ are defined only for some values of the variable x . The following general definition includes all such examples.

Definition :

Let A and B be non-empty sets. A function (or) mapping f from A into B . Written as $f : A \rightarrow B$ is a rule which assigns to each element $a \in A$, a unique element $b \in B$. The element b which corresponds in this way to a given element $a \in A$ is called the image of ' a ' under f and is written as $f(a)$. Also if $f(a) = b$ then ' a ' is called a pre-image of b under f . A is called the domain of f and $\{f(a) / a \in A\}$ is called the range of f . Two functions $f, g : A \rightarrow B$ are said to be equal if $f(x) = g(x)$ for all $x \in A$.

Examples :-

- 1) Consider the function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(x) = 2x$ clearly the domain of f is \mathbb{Z} . The range of f is given by $\{f(x) / x \in \mathbb{Z}\} = \{2x / x \in \mathbb{Z}\} = \mathbb{E}$.

The pre-image of any even integer $2n$ under f is n and any odd integer does not have a pre-image under f .

- 2) Let A be any non-empty set. $f : A \times B \rightarrow A$ defined by $f((a, b)) = a$ is called the projection onto the first coordinate projection onto the second coordinate can be similarly defined.

- 3) Let A be any non empty set. Let ρ be an equivalence relation defined on A . $f : A \rightarrow A/\rho$ defined by $f(x) = [x]$ is called the canonical map (or) natural map.

- 4) Let $E \subseteq \mathbb{R}$ the function.

$Y_E : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$Y_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Y_E is called the characteristic function on E .

Definition :

Let $f : A \rightarrow B$ be a function. The graph of f defined to be $\{(a, f(a)) / a \in A\}$. A function may be specified by its graph which is a subset of $A \times B$. Thus a function from A to B is a relation such that each element of A is related to exactly one element of B .

Remark :

A relation from A to B may fail to be a function in any one of the following ways.

- i) An element $a \in A$ may be related to more than one element in B .
- (ii) An element $a \in A$ may not be related to any element in B .

Examples :

Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$ consider the relation from A to B given by the following subset of $A \times B$.

$\{(1, 2), (1, 4), (3, 6), (5, 8), (7, 4)\}$. This is not a function from A to B . Since 1 is related to 2 and 4. Further 9 is not related to any element of B .

2) The following relation defined on R by $\{(x, \cos^{-1}x) / x \in R\}$ is not a function. Since $x=0$ is related to more than one element.

Exercises:

- 1) If A contains n elements and B contains m elements. Prove that B^A contains m^n elements.

- 2) Let $f: A \rightarrow B$ be any function define a relation ρ in A by

$$a_1 \rho a_2 \Leftrightarrow f(a_1) = f(a_2). \text{ Show that } \rho \text{ is an equivalence relation.}$$

Definitions:

A function $f: A \rightarrow B$ is one-one (injective) if distinct elements in A have distinct images in B under f . In other words f is 1-1 if $x, y \in A$ and $x \neq y$

$$\Rightarrow f(x) \neq f(y) \text{ or equivalently}$$

$$f(x) = f(y) \Rightarrow x = y.$$

The mapping f is called onto (surjective) if the range of f is equal to B . Thus f is onto every element of B has pre image in A . If $f: A \rightarrow B$ is both 1-1 and onto then f is called a bijection. In this case every element of B has exactly one pre image in A .

Examples:

1) $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(x) = 2x$ is 1-1 but not onto for $f(x) = f(y)$

$$\Rightarrow 2x = 2y$$

$$\Rightarrow x = y \text{ Hence } f \text{ is 1-1.}$$

The element of \mathbb{Z} does not have any preimage. Hence f is not onto.

2) Consider $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(x) = x + 3$

f is 1-1 For $f(x) = f(y)$

$$\Rightarrow x + 3 = y + 3$$

$$\Rightarrow x = y$$

Also any element y has $x = y - 3$ as its pre-image under f .

Hence f is onto. Hence f is a bijection.

Definition:

Let $f : A \rightarrow B$ be a function. Let $S \subseteq A$. The restriction of f to S denoted by $f|_S$ is a function from S to B defined by $f|_S(x) = f(x)$ for all $x \in S$.

Example:

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 1/(1 + x^2)$

Definition:

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions we define the composite of these functions $g \circ f : A \rightarrow C$ by the rule.

$$(g \circ f)(a) = g(f(a)) \text{ for all } a \in A.$$

Examples:

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = x^2$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ is given by $g(x) = \sin x$.

Then $(f \circ g)(x) = f(g(x)) = f(\sin x) = (\sin x)^2$ and

$$g \circ f(x) = g(f(x)) = g(x^2) = \sin x^2$$

\therefore In general $g \circ f \neq f \circ g$

Exercises:

1) If the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = \cos x$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ is given by $g(x) = x^3$ find $(g \circ f)(x)$ and $(f \circ g)(x)$ and show that they are not equal.

Thm:

If $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$ are functions then $h \circ (g \circ f) = (h \circ g) \circ f$.

Pf :

$h \circ (g \circ f)$ and $(h \circ g) \circ f$ are both functions from A to D .

Now let $x \in A$.

$$\begin{aligned} \text{Then } (h \circ (g \circ f))(x) &= h[(g \circ f)(x)] \\ &= h[g(f(x))] \end{aligned}$$

$$\text{Similarly } ((h \circ g) \circ f)(x) = h(g(f(x)))$$

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Exercise:

1) verify $(h \circ g) \circ f = h \circ (g \circ f)$ where $f : \mathbb{N} \rightarrow \mathbb{Z}$ is given by $f(x) = 5 - 3x$, $g : \mathbb{Z} \rightarrow \mathbb{R}^+$ is given by $g(x) = x^2 + 3$ and $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is given by $h(x) = \sqrt{x}$.

Thm:

Let $f : B \rightarrow C$ and $g : A \rightarrow B$ be bijections. Then $g \circ f : A \rightarrow C$ is also a bijection.

Pf:

Let $x = y \in A$.

$$(g \circ f)(x) = (g \circ f)(y)$$

$$\Rightarrow g(f(x)) = g(f(y))$$

$$\Rightarrow f(x) = f(y)$$

since f is 1-1

$$\Rightarrow x = y \text{ since } f \text{ is 1-1}$$

$g \circ f$ is 1-1

Now let $z \in C$

since $g : B \rightarrow C$ is onto there exists $y \in B$ such that $g(y) = z$. Again since $f : A \rightarrow B$ is onto. There exists $x \in A$ such that $f(x) = y$.

$$\therefore (g \circ f)(x) = g(f(x)) = g(y) = z$$

$\therefore g \circ f$ is onto. Hence $g \circ f$ is a bijection.

Thm :

Let $f : A \rightarrow B$; $g : B \rightarrow C$ be two functions. Then

- (i) gof is 1-1 $\rightarrow f$ is 1-1
(ii) gof is onto $\rightarrow g$ is onto

Pf.

(i) Let gof be 1-1 let $x, y \in A$

Then $f(x) = f(y)$

$$\Rightarrow g(f(x)) = g(f(y))$$

$$\Rightarrow (\text{gof})(x) = (\text{gof})(y)$$

$$\Rightarrow x = y \text{ (since } \text{gof} \text{ is 1-1)}$$

$\therefore f$ is 1-1

(ii) let gof be onto Let $z \in C$. Then there exists $x \in C$ such that $(\text{gof})(x) = z$.

$$\therefore g(f(x)) = z$$

z has $f(x)$ as its pre image under g .

Hence g is onto.

Exercise

Construct the functions $f : A \rightarrow B$, $g : B \rightarrow C$ such that

- (i) f is 1-1 but gof is not 1-1
(ii) f is onto but gof is not onto.
(iii) g is 1-1 but gof is not 1-1

Definitions:

Let $f : A \rightarrow B$ be a bijection. Then for each $b \in B$, exists a unique element $a \in A$ such that $f(a) = b$ we now define.

$f^{-1} : B \rightarrow A$ by $f^{-1}(b) = a$ f^{-1} is called the inverse of the function f .

Solved problems:

1) Show that $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x - 2$ is a bijection and find its inverse. Compute $f^{-1} \circ f$ and $f \circ f^{-1}$

Solution:

$$f(x) = f(y)$$

$$\Rightarrow 2x - 2 = 2y - 2$$

$$\Rightarrow x = y$$

Hence f is 1-1

Now let $y \in \mathbb{R}$ to prove that f is onto we must find $x \in \mathbb{R} \rightarrow f(x) = y$.

Now $f(x) = y$.

$$\Rightarrow 2x - 3 = y$$

$$\Rightarrow x = y + 3/2$$

Hence $y + 3/2$ is the Preimage of y under f .

$\therefore f$ is onto. Hence f is a bijection and $f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ is given by $f^{-1}(x) = (x + 3)/2$

Now $(f^{-1} \circ f)(x) : f^{-1}(f(x))$

$$= f^{-1}(2x - 3)$$

$$= (2x - 3 + 3)/2 = x$$

Similarly $(f \circ f^{-1})(x) = x$

2) Show that $f : \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{1\}$ given by $f(x) = x - 2/x - 3$ is a bijection and find its inverse.

Solution :

$$f(x) = f(y)$$

$$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$$

$$\Rightarrow xy - 3x - zy + 6 = xy - 3y - 2x + 6$$

$$\Rightarrow 3x + 2y = 2x + 3y$$

$$\Rightarrow x = y$$

Hence f is 1-1

Now let $y \in \mathbb{R} - \{1\}$ to prove that f is onto we must find $x \in \mathbb{R} - \{3\}$ such that $f(x) = y$

Now $f(x) = y$

$$\Rightarrow \frac{x-2}{x-3} = y$$

$$\Rightarrow x - 2 = xy - 3y$$

$$\Rightarrow x = \frac{2 - 3y}{1 - y}$$

Hence $\frac{2 - 3y}{1 - y}$ is the pre image of y under f .

$\therefore f$ is onto.

Hence f is a bijection and $f^{-1} : \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{3\}$

$$\text{is given by } f^{-1}(x) = \frac{2-3x}{1-x}$$

3) Show that $f: \mathbb{R} \rightarrow (0,1)$ defined by $f(x) = \frac{1}{2} \left(1 + \frac{x}{1+|x|} \right)$ is a bijection

solution:

$$\text{clearly } f(0) = \frac{1}{2}$$

$$\text{When } x > 0 : f(x) = \frac{1}{2} \left(1 + \frac{x}{1+x} \right)$$

$$\text{Hence } \left(\frac{1}{2}\right) < f(x) < 1$$

Similarly when $x < 0$;

$$f(x) = \frac{1}{2} \left(1 + \frac{x}{1-x} \right) \text{ and}$$

$$\text{hence } 0 < f(x) < \frac{1}{2}$$

Hence f maps $(0, \infty)$ to $(\frac{1}{2}, 1)$ and $(-\infty, 0)$ to $(0, \frac{1}{2})$

we first prove that $f: (0, \infty) \rightarrow (\frac{1}{2}, 1)$ is a bijection.

Let $x, y \in (0, \infty)$.

Then $f(x) = f(y)$

$$\Rightarrow \frac{1}{2} \left(1 + \frac{x}{1+x} \right) = \frac{1}{2} \left(1 + \frac{y}{1+y} \right)$$

$$\Rightarrow \frac{x}{1+x} = \frac{y}{1+y}$$

$$\Rightarrow x(1+y) = y(1+x)$$

$$\Rightarrow x = y$$

Hence f is 1-1

Now let $y \in (\frac{1}{2}, 1)$. To prove f is onto we must find $x \in (0, \infty)$ such that $f(x) = y$.

Now $f(x) = y$

$$\Rightarrow \frac{1}{2} \left(1 + \frac{x}{1+x} \right) = y$$

$$\Rightarrow \frac{x}{1+x} = 2y - 1$$

$$\Rightarrow x = \frac{2y-1}{2(1-y)}$$

Hence f is onto.

$f: (0, \infty) \rightarrow (\frac{1}{2}, 1)$ is a bijection. Similarly $f: (-\infty, 0) \rightarrow (0, \frac{1}{2})$ is also a bijection.

\therefore Hence $f: \mathbb{R} \rightarrow (0, 1)$ is a bijection.

4) Prove that a set x is infinite \Leftrightarrow there exists a bijection between x and a proper subset A of x .

Solution:

Suppose x is finite and suppose there exists a bijection $f: A \rightarrow x$ where A is a proper subset of x .

Since f is a bijection A and x have the same number of elements.

But $A \subset x$. Hence $A = x$ which is a contradiction.

Hence x is infinite.

Conversely suppose x is infinite.

Choose a sequence of distinct elements $x_1, x_2, \dots, x_n, \dots$ in x .

Let $A = x - \{x_i\}$ clearly A is a proper subset of x .

Define $f: x \rightarrow A$ by $f(x_i) = x_{i+1}$ and $f(x) = x$ if $x \neq x_i$

clearly f is a bijection from x to A .

Hence the result follows.

Exercise:

1) Which of the following functions f are bijections? If f is a bijection find f^{-1} :

a) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 3$.

b) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 5x$.

c) $f: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(x) = 5x$.

d) $f: \mathbb{C} \rightarrow \mathbb{R}$ defined by $f(x+iy) = x$.

e) $f: \mathbb{R}^* \rightarrow \mathbb{R}^*$ defined by $f(x) = 1/x$.

Definition:

Let A be any set. The function $i_A: A \rightarrow A$ defined by $i_A(x) = x$ for all $x \in A$ is called the identity function on A . Thus i_A leaves every element of A fixed.

Theorem :

Let $f: A \rightarrow A$ be any function. Then $f \circ i_A = i_A \circ f = f$.

Pf:

Let $x \in A$

Then $(f \circ i_A)(x) = f(i_A(x)) = f(x)$

Hence $f \circ i_A = f$. Similarly $i_A \circ f = f$.

Theorem :

Let $f: A \rightarrow B$ be a bijection. Then $f^{-1}: B \rightarrow A$ is also a bijection and $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

Pf:

Let $y_1, y_2 \in B$.

since $f: A \rightarrow B$ is a bijection. There exists $x_1, x_2 \in A$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$.

Now $f^{-1}(y_1) = f^{-1}(y_2)$

$$\Rightarrow x_1 = x_2$$

$$\Rightarrow f(x_1) = f(x_2)$$

$$\Rightarrow y_1 = y_2$$

Hence f^{-1} is 1-1

Now let $x \in A$. Let $f(x) = y$.

Then $f^{-1}(y) = x$. Thus every element $x \in A$ has $f(x)$ as its pre image under f^{-1} .

Hence f^{-1} is onto.

Also $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x = i_A(x)$.

Hence $f^{-1} \circ f = i_A$.

Similarly $f \circ f^{-1} = i_B$.

Theorem:

A function $f: A \rightarrow B$ is a bijection \Leftrightarrow There exists a unique function $g: B \rightarrow A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.

Pf:

Let $f: A \rightarrow B$ be a bijection. Then $f^{-1}: B \rightarrow A$ is also a bijection and $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

Now let $g: B \rightarrow A$ be any other function such that $g \circ f = i_A$ and $f \circ g = i_B$.

Let $y \in B$. Let $g(y) = x$.

Then $f(x) = f(g(y))$

$$= (f \circ g)(y)$$

$$= i_B(y) = y.$$

Hence $f^{-1}(y) = x = g(y)$

$$\therefore f^{-1} = g.$$

Conversely suppose there exists a function $g: B \rightarrow A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.

We shall prove that f is a bijection.

Let $x, y \in A$

Then $f(x) = f(y)$

$$\Rightarrow g(f(x)) = g(f(y))$$

$$\Rightarrow (go.f)(x) = (gof)(y)$$

$$\Rightarrow i_A(x) = i_A(y)$$

$$\Rightarrow x = y.$$

Hence f is 1-1.

Now let $y \in B$. then $g(y) \in A$.

$$\text{Also } f(g(y)) = (fog)(y) = i_B(y) = y.$$

$\therefore f$ is onto. Hence f is a bijection.

Theorem :

If $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijections then $(gof)^{-1} = f^{-1} \circ g^{-1}$

Pf:

Since f and g are bijections.

$(gof) : A \rightarrow C$ is a bijection.

$(gof)^{-1} : C \rightarrow A$ is a bijection.

Also $f^{-1} : B \rightarrow A$ and $g^{-1} : C \rightarrow B$ are bijections. $f^{-1} \circ g^{-1} : C \rightarrow A$ is a bijection.

Now let $z \in C$

Since g is onto There exists $y \in B$. Such that $g(y) = z$.

Since f is onto . There exists $x \in A$ such that $f(x) = y$.

Now by definition

$$g^{-1}(z) = y \text{ and } f^{-1}(y) = x$$

$$\text{Hence } (f^{-1} \circ g^{-1})(z) = f^{-1}(g^{-1}(z)) = f^{-1}(y) = x \text{ -----} > \quad (1)$$

$$\text{Also } (gof)(x) = g(f(x)) : g(y) = z.$$

$$\text{Hence } (gof)^{-1}(z) = x \text{ -----} > \quad (2)$$

From (1) and (2)

$$\text{We get } (gof)^{-1} = f^{-1} \circ g^{-1}$$

Exercise:

- 1) Prove that if $f: A \rightarrow B$ is a bijection then $(f^{-1})^{-1} = f$.
- 2) Prove that if $f: A \rightarrow B$ is 1-1 \Leftrightarrow there exists $g: B \rightarrow A$ such that $g \circ f = i_A$.

Definition :

Any function $f: A \rightarrow B$ induces two natural set mappings. If $S \subseteq A$ the image of S under f denoted by $f(S)$ is the subset of B given by $\{f(x) / x \in S\}$. Again if $T \subseteq B$ the inverse image of T under f denoted by $f^{-1}(T)$ is the subset of A given by $\{x / f(x) \in T\}$

Examples :

- 1) Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be given by $f(x) = 2x$.

Then

- a) $f(\{1, 2, 3\}) = \{2, 4, 6\}$
- b) $f^{-1}(\{1, 3, 5\}) = \emptyset$ = since there is no element $x \in \mathbb{Z}$ such that $f(x) = 1$ (or) 3 (or) 5

Thm:

Let $f: A \rightarrow B$ be a function let A_1 and A_2 be subsets of A and B_1 and B_2 subsets of B .

Then (i) $f(\emptyset) = \emptyset$

(ii) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$

(iii) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

(iv) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

(v) $f^{-1}(\emptyset) = \emptyset$

(vi) $f^{-1}(B) = A$

(vii) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

(viii) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

(ix) $f^{-1}(B_1^c) = (f^{-1}(B_1))^c$

Pf :

The Pf for (i) (ii) (iii) (iv) & (v) are obvious. Let us prove (iv) and (vii)

$$(iv) \quad x \in f(A_1 \cap A_2)$$

$$\Rightarrow x = f(y) \text{ where } y \in A_1 \cap A_2$$

$$\Rightarrow x = f(y) \text{ where } y \in A_1 \cap A_2$$

$$\Rightarrow x \in f(A_1) \text{ and } x \in f(A_2)$$

$$\Rightarrow x \in f(A_1) \cap f(A_2)$$

$$\therefore f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$$

Note:

In the above result equality need not hold good for example.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = 3$. Let $A_1 = \{1, 2\}$ $A_2 = \{3, 4\}$ clearly $A_1 \cap A_2 = \phi$
Hence $f(A_1 \cap A_2) = \phi$. But $f(A_1) \cap f(A_2) = \{3\} \cap \{3\} = \{3\}$

$$(vii) \quad x \in f^{-1}(B_1 \cup B_2) \Rightarrow f(x) \in B_1 \cup B_2$$

$$\Rightarrow f(x) \in B_1 \text{ (or) } f(x) \in B_2$$

$$\Rightarrow x \in f^{-1}(B_1) \text{ (or) } x \in f^{-1}(B_2)$$

$$\Rightarrow x \in f^{-1}(B_1) \cup x \in f^{-1}(B_2)$$

$$f^{-1}(B_1 \cup B_2) \subseteq f^{-1}(B_1) \cup f^{-1}(B_2)$$

The converse inclusion can be proved by retracing the steps Hence.

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$$

Solved Problem:

1) Let $f: x \rightarrow y$ be a function if $A \subseteq X$ and $B \subseteq Y$ show that

$$(i) \quad A \subseteq f^{-1}(f(A))$$

$$(ii) \quad f(f^{-1}(B)) \subseteq B$$

(iii) Give an example to show that equality need not hold in (i) and (ii)

(iv) In each case when will the equality hold?

Solution:

(i) Let $x \in A$

$$f(x) \in f(A)$$

$$\therefore x \in f^{-1}(f(A))$$

$$\therefore A \subseteq f^{-1}(f(A))$$

(ii) Let $y \in f(f^{-1}(B))$

\therefore There exists $x \in f^{-1}(B)$ such that $y = f(x)$

$$\text{Now } x \in f^{-1}(B) \Rightarrow f(x) \in B.$$

$$\Rightarrow y \in B.$$

$$\therefore f[f^{-1}(B)] \subseteq B.$$

(iii) Consider $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$

$$\text{Let } A = (0, 1)$$

$$\text{Then } f(A) = (0, 1)$$

and $f^{-1}[f(A)] = (-1, 1)$ which is not a subset of A .

Consider $B = (-1, 0)$ then $f^{-1}(B) = \emptyset$

$$f(f^{-1}(B)) = f(\emptyset) = \emptyset$$

$$\therefore B \text{ is not a subset of } f(f^{-1}(B))$$

(ix) We claim that the reverse inclusion is true in (i) if f is one-one.

$$\text{Let } x \in f^{-1}(f(A))$$

$$f(x) \in f(A)$$

Since f is 1-1, $x \in A$

$$\therefore f^{-1}(f(A)) \subseteq A$$

Hence equality is true in (i) if f is 1-1. We claim that the reverse inclusion is true in (ii) if f is onto.

Let $y \in B$.

Since f is onto there exists $x \in X$

such that $f(x) = y$.

$$\therefore y \in B \Rightarrow f(x) \in B$$

$$\Rightarrow x \in f^{-1}(B)$$

$$\Rightarrow f(x) \in f(f^{-1}(B))$$

$$\Rightarrow y \in f(f^{-1}(B))$$

$$\therefore B \subseteq f(f^{-1}(B))$$

Hence equality is true in (iii) if f is onto.

Exercise:

1) Give an example of a function $f : A \rightarrow B$ such that $f(A_1^2) \neq [f(A_1)]^c$

2) If $F : A \rightarrow B$ is 1-1 prove $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$

1.3. Binary operations

A binary operation is a "way of putting two things together". For example in the set N of natural numbers use an associate with any two elements $a, b \in N$ the unique element $a + b \in N$. Again with any two sets $A, B \in P(X)$ we can associate with any two elements $a, b \in N$ the unique element $a + b \in N$. Again with any two sets $A, B \in P(X)$ we can associate the set $A \cup B \in P(X)$.

Note that f in N gives rise to the function $f : N \times N \rightarrow N$ given by $(a, b) \rightarrow a + b$

Definition :

Let A be non empty set. A binary operation $*$ on A is a function $* : A \times A \rightarrow A$. The image of an ordered pair $(a, b) \in A \times A$ under $*$ is denoted by $a * b$. A set A with a binary operation $*$ defined on it is denoted by $(A, *)$

Examples:

1. Let $A = \{0, 1, 2\}$ A binary operation $*$ on A is given by $0 * 1 = 1 * 0 = 1$;
 $0 * 2, 2 * 0 = 2$

$$1 * 2 = 2 * 1 = 0$$

$$0 * 0 = 0$$

$$1 * 1 = 2$$

$$2 * 2 = 1$$

If $*$ is a binary operation on a finite set A containing n elements then the n^2 products $a * b$, $a, b \in A$ can be conveniently arranged in the form of a table containing n rows and n columns, the product $a * b$ coming in the row along 'a' and in the column along b . Thus the above binary operation on A is given by the table.

Such a table is known as a cayley table.

$*$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

2) In any non-empty set S defined by $a * b = a$ is a binary operation.

Commutativity :

In $(\mathbb{Z}, +)$, $2 + 3 = 5$ and $3 + 2 = 5$. These two equations illustrate the law $a + b = b + a$. However in $(\mathbb{Z}, -)$, $2 - 3 = -1$ and $3 - 2 = 1$. Thus we use that the law $a * b = b * a$ may be a valid equation for some binary operations and not for some others. Again consider $(\mathbb{N}, *)$ where $a * b = a$. Here $2 * 3 = 2$ and $3 * 2 = 3$.

Definition :

A binary operation $*$ on A is said to be commutative (or) to satisfy the commutative law if $a * b = b * a$ for all $a, b \in A$.

Associativity:

In $(\mathbb{Z}, +)$

$$(2 + 3) + 4 = 5 + 4 = 9$$

$$2 + (3 + 4) = 2 + 7 = 9$$

These two equations illustrate the law

$$(a + b) + c = c + (b + c) \text{ However in } (\mathbb{Z}, -)$$

$$2 - (3 - 4) = 5 - (-1) = 3$$

$$(2 - 3) - 4 = -1 - 4 = -5$$

Thus we see that the law $(a * b) * c = a * (b * c)$ may be valid expression for some binary operations and not for others.

Again consider $(\mathbb{N}, *)$ where $a * b = a^2 b$

$$2 * (3 * 4) = 3 * 36 = 144$$

$$(2 * 3) * 4 = 12 * 4 = 576$$

For union of sets, we have

$$A \cup (B \cap C) = (A \cup B) \cap C$$

Definition :

A binary operation $*$ on A is said to be associative (or) to satisfy the associative law if $(a * b) * c = a * (b * c)$ for all $a, b, c \in A$.

Generalising the associative law.

The associative law has been stated for three elements at a time however if \therefore is an associative binary operation on A and if $a, b, c, d \in A$ then

$$[(a.b).c].d = [a.(b.c)].d$$

$$= a. [b.c).d]$$

$$= (a.b) . (c.d)$$

(Justify each equality here)

This shows that all possible ways of interpreting the product $a.b.c .d$ are equivalent.

1.4. Peano's Postulates

Let N be a set of elements called natural numbers having the following properties called axioms.

$$P_1 : 1 \in N$$

P_2 : To each element n of N there corresponds a unique element n' of N called the successor of n .

P_3 : For each $n \in N$ we have $n' \neq 1$.

P_4 : If m, n are any elements of N such that $m' = n'$ then $m = n$

P_5 : (Axiom of Induction) Let S be a subset of N . such that (i) $1 \in S$ and (ii) $s \in S \Rightarrow s' \in S$ then $S = N$.

Remark

P_1 asserts that the set N is non-empty P_2 asserts the existence of a successor or 'next' element to each element of N . P_3 asserts that , is not the successor of

any natural numbers. P_3 asserts that no two distinct elements of N can have the same 'next' element (ie) $m \neq n$.

$$m' \neq n'$$

The last axiom P_5 is the basis of proofs by mathematical induction. Also from P_5 we find that starting with 1 we can reach any natural number by counting the successive numbers.

Theorem :

For each $x \in N$. We have $x' \neq x$ (ie) no element of N equals its successor.

Pf:

$$\text{Let } S = \{x \in N / x' \neq x\}$$

Claim (i) $1 \in S$ (ii) $x \in S \Rightarrow x' \in S$.

By P_1 , $1 \in N$ and by P_3 $1' \neq 1$

$\therefore 1 \in S$ suppose $x \in S$. Then $x' \neq x$. Hence by P_4 , $(x')' \neq x'$ This means $x' \in S$. Thus $x \in S \Rightarrow x' \in S$.

Hence by P_5 , $S = N$

(ie) $x' \neq x \forall x \in N$.

Theorem:

$$y \neq x + y \forall x, y \in N.$$

Pf:

$$\text{Let } S = \{y \in N / y \neq x + y \text{ for all } x \in N\}$$

By P_3 $1 \neq x'$ (ie) $1 \neq x + 1 \forall x \in N$

$$\therefore 1 \in S$$

Also $y \in S \Rightarrow y \neq x + y \forall x \in N$

$$\Rightarrow y' \neq (x + y)' \forall x \in N$$

$$\Rightarrow y' \neq x + y' \text{ (by def of +)} \quad x \in N$$

$$\Rightarrow y' \in S.$$

by P_5 ; $S = N$. Hence $y \neq x + y$ for all $x, y \in N$.

Theorem :

$$y \neq z \Rightarrow x+y \neq x+z \text{ for all } x, y, z \in \mathbb{N}.$$

Pf:

Let $y, z \in \mathbb{N}$ with $y \neq z$

$$\text{Let } S = \{x \in \mathbb{N} / x+y \neq x+z\}$$

Now $y \neq z \Rightarrow y' \neq z'$

$$\Rightarrow y+1 \neq z+1$$

$$\Rightarrow 1+y \neq 1+z$$

$$\therefore 1 \in S.$$

Suppose $x \in S$. Then $x+y \neq x+z$

$$\Rightarrow y+x \neq z+x$$

$$\Rightarrow (y+x)' \neq (z+x)'$$

$$\Rightarrow y+x' \neq z+x'$$

$$\Rightarrow x'+y \neq x'+z$$

$$\Rightarrow x' \in S$$

Hence by P_5 , $S = \mathbb{N}$.

(ie) $y \neq z \Rightarrow x+y \neq x+z \quad \forall x, y, z \in \mathbb{N}$.

1.5 Law of Trichotomy

If $x, y \in \mathbb{N}$ then exactly one of the following holds:-

(i) $x = y$ (ii) $x > y$ (iii) $x < y$.

Theorem :

The relation is greater than defined on \mathbb{N} is transitive

(ie) $x > y: y > z \Rightarrow x > z \quad \forall x, y, z \in \mathbb{N}$ (1)

Pf:

Since $x > y$ and $y > z$ There exists $u, v \in \mathbb{N}$ such that $x = y + u$ and $y = z + v$.

Hence $x : y+u = (z + v) + u$

$$= z + (v+u) = z + w$$

where $w = u+v \in \mathbb{N}$

$$\therefore x > z$$

Remark :

$$x < y : y < z \Rightarrow x < z \quad \forall x, y, z \in \mathbb{N}.$$

Theorem :

Let $x, y, z \in \mathbb{N}$ by $x > y$, then

a) $x+z > y+z$ (monotone property for addition)

b) $xz > yz$ (monotone property for multiplication)

Pf:

a) since $x > y$, there exists $u \in \mathbb{N}$ such that $x = y+u$.

$$\text{Hence } x+z = (y+u) + z = y + (u+z)$$

$$= y + (z+u)$$

$$= (y+z) + u$$

$$\therefore x + z > y + z$$

$$\text{b) } xz = (y+u)z = yz + uz = yz + v$$

$$\text{where } v = uz \in \mathbb{N}$$

$$\therefore x, z > y, z$$

Note :

$$(i) \quad x < y \Rightarrow x + z < y + z$$

$$(ii) \quad x < y \Rightarrow xz < yz \quad \forall x, y, z \in \mathbb{N}.$$

Definition :

Let A be a non-empty subset of \mathbb{N} . Then a natural number $\ell \in A$ is said to be a least (or) first element of A if $\ell \leq a \quad \forall a \in A$.

Well - ordering principles

Every non-empty subset of \mathbb{N} has a least (or a first) element.

Pf:

Let A be a non-empty subset of \mathbb{N} .

Let $S = \{x \in \mathbb{N} / x \leq a \text{ for all } a \in A\}$

$1 \leq x$ for any $x \in \mathbb{N}$ In particular

$1 \leq a \quad \forall a \in A \quad \therefore 1 \in S.$

We have to show that for any

$y \in A, y+1 \notin S.$

Suppose $y+1 \in S$. Then $y+1 \leq a \quad \forall a \in A.$

In particular $y+1 \leq y$ which is not true

$\therefore y+1 \notin S$. Thus S is a non empty proper subset of \mathbb{N} .

Again if $\forall s \in S$: we have $s+1 \in S$. Then by the axiom of induction. We get $S = \mathbb{N}$ which is a contradiction. Therefore there exists a natural number ℓ such that $\ell \in S$ but $\ell+1 \notin S$.

We have to show that ℓ is a least element of A .

Now $\ell \in S \Rightarrow \ell \leq a \quad \forall a \in A$ Also $\ell \notin A$. then $\ell < a \quad \forall a \in A : \ell+1 \leq a \quad \forall a \in A.$

(ie) $\ell+1 \in S$ which is a contradiction therefore $\ell \in A$. Thus $\ell \in A$ and $\ell \leq a \quad \forall a \in A$

(ie) ℓ is a least element of A .

Note:

An example might be in order.

$A = \{3, 9, 14, 25, 98\}$ is a non-empty proper subset of \mathbb{N} the number 3 is the least element of A .

1.6 Mathematical induction

First Principle of Induction

Theorem :

Let $\{P(n) / n \in \mathbb{N}\}$ be a set of statements one for each for each natural number n .

If (i) $P(1)$ is true and

(ii) $P(k)$ is true

$\Rightarrow P(K+1)$ is true

then $P(n)$ is true for all natural numbers n .

Pf:

Let $S = \{s / S \in \mathbb{N} / P(S) \text{ is true}\}$

By (i) $P(1)$ is true and hence $1 \in S$; $s \in S$

$\Rightarrow P(s)$ is true \Rightarrow

$P(S+1)$ is true (by ii)

$\Rightarrow S+1 \in S$ $s' \in S$

\therefore by the axiom of induction P_5 .

$S = \mathbb{N}$ (ie) $P(n)$ is true for all natural numbers n .

Many theorems are proved more easily by induction method than by direct methods. The proof consists of two stages namely (i) establishing that the statement $P(n)$ under consideration is true for $n = 1$ (ie) $P(1)$ is true (ii) assuming that $P(K)$ is true (K is any natural number) and showing that $P(K+1)$ is true.

Proof by the method of induction can be compared with climbing an infinite ladder to climb such a ladder we have to climb first rung of the ladder and having climbed any one rung the next rung also should be climbed.

Solved Problems:

$$1) 1+2+\dots+n = \frac{n(n+1)}{2}$$

Pf:

$$\text{Let } P(N) = 1+2+\dots+n = \frac{n(n+1)}{2}$$

Stage - 1

When $n = 1$ LHS = 1

$$\text{and RHS} = \frac{1(1+1)}{2} = 1$$

$\therefore P(1)$ is true.

Stage - 2 :-

Assuming that $p(K)$ is true.

$$\text{(i.e.) } 1 + 2 + \dots + K = \frac{K(K+1)}{2}$$

This is known as induction hypothesis

To prove that $p(K+1)$ is true.

$$\text{(i.e.) } 1 + 2 + \dots + (K+1) = \frac{(K+1)(K+1) + 1}{2}$$

Now $1 + 2 + \dots + K + 1$

$$= (1 + 2 + \dots + K) + K + 1$$

$$= \frac{K(K+1)}{2} + (K+1) \text{ by induction hypothesis}$$

$$= \frac{(K+1)(K+2)}{2}$$

$$= \frac{(K+1)(K+1+1)}{2}$$

$\therefore p(K+1)$ is true

Thus by the (first Principle) of induction $p(n)$ is true for all natural numbers n .

$$\text{(i.e.) } 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

2) Show that by the method of induction that $n^2 > 3(n+1)$ for $n \geq 4$ and hence that $3n > n^3$ for $n \geq 4$.

Pf :-

(i) Say $p(n) : n^2 > 3(n+1)$ for $n \geq 4$.

When $n=4$; $n^2=16$ and $3(n+1) = 15$

Since $16 > 15$; $p(4)$ is true.

Assume that $p(K)$ is true ($K \geq 4$)

$$(i.e.) k^2 > 3(K+1) \text{ ----- (1)}$$

To show that $p(K+1)$ is true.

$$(i.e.) (K+1)^2 > 3(K+1+1)$$

$$\text{Now } (K+1)^2 = K^2 + 2K + 1$$

$$> 3(K+1) + 2K + 1 \text{ from (1)}$$

$$= 5K + 4.$$

$$> 3K + 6 \quad (\because k \geq 4)$$

$$(i.e.) (K+1)^2 > 3(K+2)$$

Hence $p(k+1)$ is true. By the principle of Induction it follows that $p(n)$ is true for $n \geq 4$.

(ii) Again Let $p(n) : 3^n > n^3$ for $n \geq 4$.

$$\text{When } n = 4; 3^n = 3^4 = 81 \text{ and } n^3 = 4^3 = 64$$

Since $81 > 64$; $p(1)$ is true

Suppose that $p(k)$ is true ($k \geq 4$)

$$(i.e.) 3^k > k^3 \text{ ----- (2)}$$

To show that $3^{k+1} > (k+1)^3$

$$\text{Now } 3^{k+1} = 3 \cdot 3^k > 3 \cdot k^3 \text{ From (2)}$$

$$\text{But } n^2 > 3(n+1) \text{ for } n \geq 4$$

$$\therefore K^2 > 3(k+1) \text{ and hence}$$

$$K^3 > 3K(K+1) = 3K^2 + 3K$$

$$\therefore K^3 = K^3$$

$$K^3 > 3K^2 + 3K \therefore K^3 > K^2$$

$$K^3 > 1 \quad (K \geq 4)$$

Adding $3k^3 > k^3 + 3k^2 + 3k + 1 = (k+1)^3$.

$\therefore 3k+1 > (k+1)^3$ (i.e.) $p(k+1)$ is true.

By the principle of induction it follows that $p(n)$ is true for $n \geq 4$.

Second Principle of induction :-

Theorem :-

Let $\{ p(n) / n \in \mathbb{N} \}$ be a set of statements one for each natural number n .

If (i) $P(1)$ is true.

(ii) if for each natural

number k , $p(m)$ is true for all $m < k$ implies that $p(k)$ is true then $p(n)$ is true for all natural numbers n .

Pf :-

Let $S = \{ n \in \mathbb{N} / p(n) \text{ is false} \}$

Since $p(1)$ is true $1 \notin S$. We have to show that $S = \emptyset$

Suppose $S \neq \emptyset$. Then S is a non-empty subset of \mathbb{N} and hence by the well ordering principle S has a least element ℓ . (i.e.) $\ell \in S$ and $\ell \leq n \forall n \in S$ observe that $\ell \neq 1$ ($\ell \notin S$)

If $m < \ell$ then $m \notin S$ ($\because \ell$ is the least element of S) Hence $p(m)$ is true for each $m < \ell$. Thus by hypothesis (ii) it follows that $p(\ell)$ is true. This means that $\ell \notin S$ which is a contradiction. $\therefore S = \emptyset$. Thus $p(n)$ is true for all natural numbers n .

UNIT - 2

2.1. Sub groups

Definition :

Let G be a set with a binary operation \star defined on it. Let $S \leq G$. Then if for each $a, b \in S$ $a \star b$ (Computed in G) is in S . We say that S is closed with respect to the binary operation " \star ".

Example :

1. $(\mathbb{Z}, +)$ is a group. The set \mathbb{E} of all even integers is closed under $+$ and further $(\mathbb{E}, +)$ is itself a group.
2. The set G of all non singular 2×2 matrices form a group under matrix multiplication. Let H be the set of all matrices of the form

$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. H is subset of G . Also G, H itself is a group under matrix multiplication.

Def inition :

A subset H of a group G is called a subgroup of G if H forms a group with respect to the binary operation in G .

Example :

1. Let G be any group. Then $\{e\}$ and G are subgroups of G . They are called improper subgroups of G .
2. $(\mathbb{Q}, +)$ is a subgroup of $(\mathbb{R}, +)$ and $(\mathbb{R}, +)$ is a subgroup of $(\mathbb{C}, +)$.

Thm :

Let H be a subgroup of G . Then

- a. The identity element of H is the same as that of G .
- b. For each $a \in H$, the inverse of a in H is the same as the inverse of a in G .

Proof :

- a. Let e and e' be the identities of G and H respectively.

Let $a \in H$, Now $e'a = a$ (Since e' is the identity of H)

$= ea$ (Since e is the identity of G and $a \in G$)

$\therefore e'a = ea$

$\therefore e' = e$ (by cancellation law).

b. Let a and a'' be the inverse of a in G and H respectively. Since by (a). G and H have the same identity element e , we have

$a'a = e = a''a$. Hence by cancellation law,

$a' = a''$.

Intersection of Subgroups :

Thm : 1

If H_1 and H_2 are two subgroups of a group G then $H_1 \cap H_2$ is also subgroups of G .

Proof :

Let H_1 and H_2 be any two subgroups of G . Then $H_1 \cap H_2 \neq \phi$. Since at least the identity element e is common to both H_1 and H_2 .

In order to prove that $H_1 \cap H_2$ is a subgroup. It is sufficient to prove that $a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$.

Now $a \in H_1 \cap H_2 \Rightarrow a \in H_1 \text{ \& } a \in H_2$.

$b \in H_1 \cap H_2 \Rightarrow b \in H_1 \text{ \& } b \in H_2$.

But H_1, H_2 are subgroups. Therefore

$a \in H_1, b \in H_1 \Rightarrow ab^{-1} \in H_1$.

$a \in H_2, b \in H_2 \Rightarrow ab^{-1} \in H_2$.

Finally $ab^{-1} \in H_1, ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$.

\therefore We have show that

$a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$.

Hence $H_1 \cap H_2$ is a subgroup of G .

Thm :

Arbitrary intersection of subgroups.

(i.e) the intersection of any family of subgroups of a group is a subgroups.

Proof :

Let G be a group. Let $\{H_t / t \in T\}$ be any family of subgroups of G . Here T is an index set and is such $\forall t \in T. H_t$ is a subgroup of G .

Let $H = \bigcap_{t \in T} H_t = \{x \in G; x \in H_t \forall t \in T\}$.

Be the intension of this family of subgroups of G . Then to prove that H is also a subgroup of G .

Obviously $H \neq \emptyset$; Since at least the identity element e is in $H_t \forall t \in T$.

Now Let a, b be any two elements of H .

Then $a \in \bigcap_{t \in T} H_t \Rightarrow a \in H_t \forall t \in T$

and $b \in \bigcap_{t \in T} H_t \Rightarrow b \in H_t \forall t \in T$

But $\forall t \in T; H_t$ is a subgroups of G . Therefore $a \in H_t, b \in H_t \Rightarrow ab^{-1} \in H_t \forall t \in T$.

Consequently $ab^{-1} \in \bigcap_{t \in T} H_t$

\therefore We have shown that $a, b \in \bigcap_{t \in T} H_t$.

$\therefore \bigcap_{t \in T} H_t$ is a subgroup of G .

Note : 1

$H_1 \cap H_2$ is the largest subspace of G which is contained in H_1 as well as in H_2 . Therefore $H_1 \cap H_2$ is the largest subgroup of G contained in H_1 and H_2 . By largest use mean that it is contained in H_1 and H_2 and contains every subgroup of G contained in both H_1 and H_2 .

Note : 2

The union of two subgroups is not necessarily a subgroup.

Proof :

For example, Let G be the additive group of integers.

Then $H_1 = \{\dots -6, -4, -2, 0, 2, 4, 6, 8, \dots\}$

$H_2 = \{\dots -12, -9, -6, -3, 0, 3, 6, 9, 12, \dots\}$

are both subgroups of G .

We have $H_1 \cup H_2 = \{\dots -4, -3, -2, 0, 2, 3, 4, 6, \dots\}$

Obviously $H_1 \cup H_2$ is not closed with respect to addition as $2 \in H_1 \cup H_2$

$$3 \in H_1 \cup H_2$$

But $2 + 3 = 5 \notin H_1 \cup H_2$. Therefore

$H_1 \cup H_2$ is not a subgroup of G .

But $H_1 \cap H_2 = \{-18, -12, -6, 0, 6, 12, 18, \dots\}$ is a subgroup of G .

If we take the subgroup

$H_3 = \{\dots -8, -4, 0, 4, 8, \dots\}$ of G , then $H_1 \cup H_3 = H_1$ and H_1 is a subgroup of G . We shall prove in one of the following examples that the union of two subgroups is a subgroup iff one is contained in the other.

Example :

1. Let 'a' be an element of a group G . The set $H = \{a^n; n \in \mathbb{I}\}$ of all integral powers of 'a' is a subgroup of G .

Proof :

We have $a \in G$. To prove that $H = \{\dots a^{-3}, a^{-2}, a^{-1}, a^0, a^1, \dots\}$ is a subgroup of G .

Let a^r, a^s be any two elements of H . Where r and s are some integers. The inverse of a^s in G is a^{-s} . Now

$$(a^r)(a^s)^{-1} = a^r a^{-s} = a^{r-s} \in H.$$

Since $r - s$ is also some integer.

Therefore H is a subgroup of G .

2. Prove that those elements of a group G which commute with the square of a given element b of G form a subgroup, H of G and those which commute with 'b' itself form a subgroup of H .

Proof :

Let $H = \{x \in G; x b^2 = b^2 x\}$. Then to prove that H is a subgroup of G . We see that H is not empty because $eb^2 = b^2 = b^2e \Rightarrow e \in H$.

Now Let $x_1, x_2 \in H$

$$\text{Then } x_1 b^2 = b^2 x_1 \text{ and } x_2 b^2 = b^2 x_2$$

First we shall show that, $x_2^{-1} \in H$, we have

$$x_2 b^2 \Rightarrow b^2 x_2 \Rightarrow x_2^{-1} (x_2 b^2) x_2^{-1} = x_2^{-1} (b^2 x_2) x_2^{-1}.$$

$$\Rightarrow b^2 x_2^{-1} = x_2^{-1} b^2.$$

$$\Rightarrow x_2^{-1} \in H.$$

Now we shall show that $x_1 x_2^{-1} \in H$.

$$\text{We have } x_1 x_2^{-1} b^2 = x_1 b^2 x_2^{-1} \quad (\because b^2 x_2^{-1} = x_2^{-1} b^2)$$

$$= b^2 x_1 x_2^{-1} \quad (\because x_1 b^2 = b^2 x_1)$$

$$\therefore x_1 x_2 \in H.$$

Thus $x_1 x_2 \in H \Rightarrow x_1 x_2^{-1} \in H \Rightarrow H$ is a subgroup of G .

Let $N = \{y \in G / yb = by\}$; we have $yb = by$

$$\Rightarrow (yb) b = (by) b$$

$$\Rightarrow yb^2 = b(yb)$$

$$\Rightarrow yb^2 = b(by)$$

$$\Rightarrow yb^2 = b^2 y$$

$$\text{Thus } y \in N \Rightarrow y \in H$$

$$\therefore N \leq H.$$

Now to prove that N is a subgroup of H . Obviously N is not empty. Since at least $e \in N$. Let $y_1, y_2 \in N$. Then $y_1 b = by_1$ and $y_2 b = by_2$

$$\text{We have } y_2 b = by_2.$$

$$\Rightarrow y_2^{-1} (y_2 b) y_2^{-1} = y_2^{-1} (by_2) y_2^{-1}.$$

$$\Rightarrow by_2^{-1} = b_2^{-1} b$$

$$\text{Now } y_1 y_2^{-1} b = y_1 by_2^{-1} = by_1 y_2^{-1}.$$

$$\therefore y_1 y_2^{-1} \in N.$$

$$\therefore y_1, y_2 \in N \Rightarrow y_1 y_2^{-1} \in N$$

Hence N is a subgroup of H .

3. Show that the union of two subgroups is a subgroup if and only if one is contained in the other.

Proof :

Suppose H_1 and H_2 are two subgroups group G . Let $H_1 \leq H_2$ (Or) $H_2 \leq H_1$.

But H_1, H_2 are subgroups.

$\therefore H_1 \cup H_2$ is also a subgroup.

Conversely suppose $H_1 \cup H_2$ is a subgroup.

To prove that $H_1 \leq H_2$ (or) $H_2 \leq H_1$.

Let us assume that H_1 is not a subset of H_2 . Now H_1 is not a subset of $H_2 \Rightarrow \exists a \in H_1$ and $a \notin H_2$ ----- (1)

and H_2 is not a subset of $H_1 \Rightarrow \exists b \in H_2$ & $b \notin H_1$ ----- (2)

from (1) and (2) we have

$$a \in H_1 \cup H_2.$$

$$b \in H_1 \cup H_2.$$

Since $H_1 \cup H_2$ is a subgroup. $\therefore ab = c$ (Say)

is also an element of $H_1 \cup H_2$.

But $ab = c \in H_1 \cup H_2$

$\Rightarrow ab = c \in H_1$ (or) H_2

Suppose $ab = c \in H_1$.

$$b = a^{-1}c \in H_1 \quad (\because H_1 \text{ is a subgroup})$$

$$\therefore a \in H_1 \Rightarrow a^{-1} \in H_1.$$

But from (2) we have $b \notin H_1$.

\therefore We get a contradiction.

Again suppose $ab = c \in H_2$.

Then $a = cb^{-1} \in H_2$ ($\because H_2$ is a subgroup, $\therefore b \in H_2 \Rightarrow b^{-1} \in H_2$.)

But from (1) we have $a \notin H_2$.

\therefore Thus here also we get a contradiction.

Hence either $H_1 \leq H_2$ (or) $H_2 \leq H_1$.

The Centre of a group :

Definition :

The set Z of all self-conjugate elements of a group G is called the centre of G . Symbolically,

$$Z = \{Z \in G; Zx = xZ \forall x \in G\}$$

Thm :

The centre Z of a group G is a normal subgroup of G .

Proof :

$$\text{We have } Z = \{Z \in G; Zx = xZ \forall x \in G\}$$

First we shall prove that Z is a subgroup G .

Let $Z_1, Z_2 \in Z$. Then $Z_1x = xZ_1$ and $Z_2x = xZ_2$ for all $x \in G$.

$$\text{We have } Z_2x = xZ_2 \forall x \in G.$$

$$\Rightarrow Z_2^{-1}(Z_2x)Z_2^{-1} = Z_2^{-1}(xZ_2)Z_2^{-1}.$$

$$\Rightarrow xZ_2^{-1} = Z_2^{-1}x \forall x \in G.$$

$$\Rightarrow Z_2^{-1} \in G.$$

$$\text{Now } (Z_1Z_2^{-1})x = Z_1(Z_2^{-1}x)$$

$$= Z_1(xZ_2^{-1})$$

$$= (Z_1x)Z_2^{-1}$$

$$= (xZ_1)Z_2^{-1}$$

$$= x(Z_1Z_2^{-1})$$

$$\therefore Z_1Z_2^{-1} \in Z$$

$$\text{Thus } Z_1, Z_2 \in Z \Rightarrow Z_1Z_2^{-1} \in Z$$

$\therefore Z$ is a subgroup of G .

Now we shall show that Z is a normal subgroup of G . Let $x \in G$ and $Z \in Z$.

$$\text{Then } xZx^{-1} = (xz)x^{-1} = (zx)x^{-1} = Z \in Z.$$

$$\text{Thus } x \in G, Z \in Z \Rightarrow xZx^{-1} \in Z.$$

$\therefore Z$ is a normal subgroup of G .

Thm :

$a \in Z$ iff $N(a) = G$. If G is finite. $a \in Z$ if and only if $O\{CN(a)\} = O(G)$.

Proof :

Let $a \in Z$. Then by def of Z , we have $ax = xa \forall x \in G$.

Also $N(a) = \{x \in G, ax = xa\}$

Now $a \in Z \Leftrightarrow ax = xa \forall x \in G$.

$\Leftrightarrow x \in N(a) \forall x \in G$.

$\Leftrightarrow N(a) = G \{ \because N(a) \leq G \text{ and each element of } G \text{ is in } N(a) \}$

If the group G is finite, then $N(a) = G$

$\Leftrightarrow O(G) = O\{N(a)\}$.

\therefore If the group G is finite, then $a \in Z \Leftrightarrow$

$O\{N(a)\} = O(G)$

Normalizer of an element of a group :

Definition :

If $a \in G$, then $N(a)$ the normalizer of 'a' in G is the set of all those elements of G which commute with 'a'. Symbolically $N(a) = \{x \in G / ax = xa\}$.

Theorem :

The normalizer $N(a)$ of $a \in G$ is a subgroup of G .

Proof :

We have $N(a) = \{x \in G / ax = xa\}$.

Let $x_1, x_2 \in N(a)$ Then $ax_1 = x_1a$; $ax_2 = x_2a$.

First we show that $x_2^{-1} \in N(a)$.

We have $ax_2 = x_2a$.

$$\Rightarrow x_2^{-1}(ax_2)x_2^{-1} = x_2^{-1}(x_2a)x_2^{-1}.$$

$$\Rightarrow x_2^{-1}a = ax_2^{-1} \Rightarrow x_2^{-1} \in N(a).$$

Now we shall show that $x_1x_2^{-1} \in N(a)$.

$$\text{We have } a(x_1x_2^{-1}) = (ax_1)x_2^{-1} = (x_1a)x_2^{-1}.$$

$$\Rightarrow x_1 (ax_2^{-1}) = x_1 (x_2^{-1}a) = (x_1x_2^{-1})_g.$$

$$\therefore x_1x_2^{-1} \in N(a)$$

$$\text{Thus } x_1, x_2 \in N(a) \Rightarrow x_1, x_2^{-1} \in N(a)$$

$$\therefore N(a) \text{ is a subgroup of } G.$$

Note 1 :

It should be noted that $N(a)$ is not necessarily a normal subgroup of G .

Note 2 :

$$\text{Since } ex = xe \quad \forall x \in G. \quad \therefore N(e) = G.$$

Note 3 :

$$\text{If } G \text{ is an abelian group and } a \in G \text{ then } xa = ax \quad \forall x \in G. \quad \therefore N(a) = G.$$

Thm: Let a be any element of a group G . Then two elements $x, y \in G$ give rise to the same conjugate of a if and only if they belong to the same right coset of the normalizer of ' a ' in G .

Proof :

We have

$x, y \in G$ are in the same right coset of $N(a)$ in G .

$$\Leftrightarrow N(a)x = N(a)y \quad [\because x \in N(a)x; y \in N(a)y. \text{ if } H \text{ is a subgroup, then } x \in Hx]$$

$$\Leftrightarrow xy^{-1} \in N(a) \quad [\because \text{if } H \text{ is a subgroup, then } Ha = Hb \Leftrightarrow ab^{-1} \in H.]$$

$$\Leftrightarrow axy^{-1} = xy^{-1}a \quad Ha = Hb \Leftrightarrow ab^{-1} \in H.$$

$$\Leftrightarrow x^{-1}ax = y^{-1}ay.$$

$$\Leftrightarrow x, y \text{ give rise to the same conjugate of 'a'.$$

Hence the result.

2.2. Permutations

Definition :

A one-one mapping of a finite set onto itself is called a permutation.

Since it is one-one on to it is invertible. The number of elements in the finite set is known as the degree of the permutation.

Clearly a set consisting of three elements will have $3! (=6)$ permutations. We denote this set of permutations by P_3 and call it the symmetric set of

transformations of degree three. In the same way, we denote by P_n , the set of all permutations of a set containing n elements.

Symbolical Representation of a Permutation :

Let $S = \{a_1, a_2, \dots, a_n\}$. Where $a_i \neq a_j$ for $i \neq j$.

Let f be a transformation on S_1 so that f is a permutation on S . We now introduce a two line notation for the permutation.

$$f = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$$

Similarly if C_1, \dots, C_n is another arrangement of the elements of S_1 we write.

$$g = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ c_1 & c_2 & \dots & c_n \end{pmatrix}$$

It is obvious from the definition that a permutation can be written in several ways by interchanging vertical lines (Columns) in a manner that the corresponding elements above and below remain unchanged.

Thus the Permutation :

$$f = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$$

can be written as

$$\begin{pmatrix} a_1 & a_3 & a_1 & \dots & a_n & a_{n-1} \\ b_1 & b_3 & b_1 & \dots & b_n & b_{n-1} \end{pmatrix}$$

$$(Or) \text{ as } \begin{pmatrix} a_1 & a_3 & a_4 & a_2 & \dots & a_n \\ b_1 & b_3 & b_4 & b_2 & \dots & b_n \end{pmatrix}$$

$$(Or) \text{ as } \begin{pmatrix} a_3 & a_1 & a_4 & a_n & \dots & a_2 \\ b_3 & b_1 & b_4 & b_n & \dots & b_2 \end{pmatrix} \text{ etc.}$$

Product of Two Permutations :

Let the two permutations on the set

$$S = (a_1 \ a_2 \ a_3 \ \dots \ a_n)$$

$$f = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ b_1 & b_2 & b_3 & \dots & b_n \end{pmatrix}$$

$$g = \begin{bmatrix} b_1 & b_2 & b_3 & \dots & b_n \\ c_1 & c_2 & c_3 & \dots & c_n \end{bmatrix}$$

By the product fog_1 we shall mean that t and g performed in that order.

(i.e.) first operation by f and then by g .

$$\text{Then } fog = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ c_1 & c_2 & c_3 & \dots & c_n \end{bmatrix}$$

For f replaces a_1 by b_1 and then g replaces b_1 by c_1 so that fog replaces a_1 by c_1 so that fog replaces a_1 by c_1 .

Similarly fog replaces a_2 by c_2 , a_3 by c_3 a_n by c_n .

Clearly fog is also a permutation on S .

Thus for all $f, g \in P_n$; $fog \in P_n$.

Example : 1

$$\text{Let } f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$$

$$g = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{bmatrix}$$

Since the order of columns of any permutation is immaterial. i.e. may rewrite g as,

$$g = \begin{bmatrix} 2 & 3 & 4 & 5 & 1 \\ 3 & 2 & 5 & 4 & 1 \end{bmatrix}$$

the lower line of f .

in which the upper line of g is

$$\begin{aligned} fog &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 & 5 & 1 \\ 3 & 2 & 5 & 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{bmatrix} \end{aligned}$$

Identity Permutation :

We denote the identity permutation by I .

$$\text{Then } I = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & a_2 & \dots & \dots & a_n \end{bmatrix}$$

Inverse Permutation :

Since a permutation is a one-one onto mapping. It is inversible (i.e.) every permutation f has an inverse denoted as usual by f^{-1} .

$$\therefore \text{ if } f = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$$

$$\text{Then } f^{-1} = \begin{pmatrix} b_1 & b_2 & \dots & b_n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

Cyclic Permutations :

Definition:

A permutation which replace n objects cyclically is called a cyclic (or circular) permutation of degree n .

Thus the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \text{ as } (3.45)$$

Where the cycle $(3 \ 4 \ 5)$ is interpreted to mean that 1 and 2 the missing symbols are unchanged while 3 is replaced by 4, 4 by 5 and 5 by 3.

As another example, let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 1 & 8 & 9 & 7 \end{pmatrix}$$

$$f = (5) \circ (1 \ 2 \ 3 \ 4 \ 6) \circ (7 \ 8 \ 9)$$

$$\text{(Or) simply } f = (1 \ 2 \ 3 \ 4 \ 6) \circ (7 \ 8 \ 9)$$

(i.e) see that every permutation can be expressed as a composite of cycles.

Definition :

The number of objects permuted by a cycle is called its length.

Disjoint cycles :

Two cycles are said to be disjoint if they have no symbols in common.

Thus $(1 \ 3 \ 4)$ and $(2 \ 5)$ are disjoint but $(1 \ 3 \ 4)$ and $(2 \ 3 \ 5)$ are not disjoint.

Powers of a cyclic permutation :

Let $f = (1 \ 2 \ 3 \ \dots \ n)$ Now f moves every symbol to the next place on the circumference of circle to obtain f^3 we move every symbol two places along; to obtain f^4 we move every symbol three places along and so on.

Example : 2

$$\text{If } f = (1 \ 2 \ 3 \ 4 \ 5 \ 6)$$

$$f^2 = (1 \ 3 \ 5) (2 \ 4 \ 6)$$

$$f^3 = (1 \ 4) (2 \ 5) (3 \ 6)$$

$$f^4 = (1 \ 5 \ 3) (2 \ 6 \ 4)$$

$$f^5 = (1 \ 6 \ 5 \ 4 \ 3 \ 2)$$

$$f^6 = (1) (2) (3) (4) (5) (6) = I$$

$$f^9 = f^3 = (1 \ 4) (2 \ 5) (3 \ 6)$$

$$f^{32} = f^{30} \circ f^2 = f^2 = (1 \ 3 \ 5) (2 \ 4 \ 6)$$

Transpositions :

Definitions :

A cycle of length two is called a transposition.

Thus a transposition is a cycle of the form (a_i, a_j) in which the symbols a_i, a_j are interchanged and other symbols remain unchanged.

Notice that every cycle can be expressed as a composite of transpositions. For example $(a \ b \ c \ d) = (a \ b) \circ (a \ c) \circ (a \ d)$

$$\text{Also } (a \ b \ c \ d) = (a \ b) \circ (b \ c) \circ (c \ d) \circ (d \ b) \circ (a \ c)$$

It should be noted that the cycle $(a \ b \ c \ d)$ has been expressed as a composite of transposition in two different ways. In both of them the number of transposition is odd.

Thm :

If a permutation P is a product of S transpositions and also a product of r transposition then $r \equiv s \pmod{2}^*$, (ie) r and s are either both odd (or) both even.

Proof :

In our proof, we use the alternative polynomial A . In distinct symbols x_1, x_2, \dots, x_n . It is defined as the product of the $\frac{1}{2}n(n-1)$ factors of the form $x_i - x_j$ where $i < j$.

$$\text{Thus } A = \prod_{i < j = 1}^n x_i - x_j$$

$$= \{(x_1-x_2) (x_1-x_3) (x_1-x_4) \dots (x_1-x_n) \}$$

$$(x_2-x_3) (x_2-x_4) \dots (x_2-x_n) (x_3-x_4)$$

$$(x_3-x_5) \dots (x_{n-1}-x_n)$$

Consider now any permutation P on 'n' symbols 1, 2, 3.....n. By AP we mean the polynomial obtained by permuting the subscripts 1, 2,n of the x_g as prescribed by P.

For example taking $n=4$, we have

$$A = (x_1-x_2) (x_1-x_3) (x_1-x_4) (x_2-x_3) (x_2-x_4) (x_3-x_4)$$

and if $P = (1 \ 3 \ 4 \ 2)$, then

$$AP = (x_3-x_1) (x_3-x_4) (x_3-x_2) (x_1-x_4) (x_1-x_2) (x_4-x_2)$$

In Particular if $P = (1 \ 2)$ we have

$$AP = (x_2-x_1) (x_2-x_3) (x_2-x_4) (x_1-x_3) (x_1-x_4) (x_3-x_4)$$

$$= -A.$$

This shows that the effect of a transposition on A is to change the sign of A.

In general a transposition (i, j), $i < j$ has the following effects on A :

- (i) Any factor which involves neither the suffix i nor j remains unchanged.
- (ii) The single factor (x_i-x_j) changes its sign.
- (iii) The remaining factors which involve either the suffix i (or) j but not both can be grouped into pairs of products.
 $\pm (x_m-x_i) (x_m-x_j)$ where $m \neq i$ (or) j and such a product remains unaltered when x_i and x_j are interchanged.

Hence the net effect of the transposition (i, j) on A is to change its sign (i.e.) operated upon by a transposition (i,j) gives $-A$.

Now the permutation P considered as a product of S transpositions when operated upon A gives $(-1)^S A$ so that $AP = (-1)^S A$ and considered as a product of t transpositions gives $(-1)^t A$.

$$\text{So that } AP = (-1)^t A$$

$$\text{Hence } (-1)^S A = (-1)^t A$$

$$(-1)^S = (-1)^t \text{ ----- (1)}$$

Now (1) will hold only if S and t are either both even (or) both odd.

Hence the result.

Even and Odd Permutations :

Definition :

A Permutation is called even (or) odd according as it can be expressed as a composite of even (or) odd number of transpositions.

Thm :

Of the $n!$ permutations on ' n ' symbols $\frac{1}{2}n!$ are even permutations and $\frac{1}{2}n!$ are the odd permutations.

Proof :

Let the even permutations be e_1, \dots, e_m and the odd permutations be o_1, o_2, \dots, o_k .

There $m+k = n!$.

Now let t be any transposition. Since t is evidently an odd permutation we see that te_1, te_2, \dots, te_m are odd permutations and that to_1, to_2, \dots, to_k are even permutations. Since an odd permutation is never an even permutation. We have,

$$te_i \neq to_j.$$

for any $i = 1, 2, \dots, m, j = 1, 2, \dots, k$ further more if $te_i = te_j$.

Then $e_i = e_j$ by cancellation law.

Similarly $to_i \neq to_j$ if $i \neq j$.

It follows that all of the m even permutations must appear in the list to_1, to_2, \dots, to_k .

Which are all distinct so that their number is m . Similarly all of the k odd permutations must be in the list te_1, te_2, \dots, te_m . Which are all distinct as shown. Above so that their number is K . Hence $m = K = \frac{1}{2} n!$.

Remark : 1

A cycle containing an odd number of symbols is an even permutation whereas a cycle containing an even number of symbols is an odd permutation. Since a permutation on n symbols can be expressed as a product of $(n-1)$ transposition.

Remark : 2

Inverse of an even permutation is an even permutation and the inverse of an odd permutation is an odd permutation.

Remark : 3

Product of two permutations is an even permutations if either both the permutations one even (or) both are odd and the product is an odd permutation if one permutation is odd and the other even.

Associativity of Composites of permutations :

Thm :

Multiplication of permutation is associative.

Proof :

Let f, g, h be any three permutations on the set $S = \{a_1, a_2, \dots, a_n\}$. After a suitable arrangement we can write these permutations as

$$f = \begin{pmatrix} A \\ B \end{pmatrix} \quad g = \begin{pmatrix} B \\ C \end{pmatrix} \quad h = \begin{pmatrix} C \\ D \end{pmatrix}$$

Where A, B, C, D denote arrangements of the n symbols in S of the form.

$$A = a_1, a_2, \dots, a_n.$$

$$B = b_1, b_2, \dots, b_n.$$

$$C = c_1, c_2, \dots, c_n.$$

$$D = d_1, d_2, \dots, d_n.$$

$$\text{Then } (fog) oh = \left(\begin{pmatrix} A \\ B \end{pmatrix} \circ \begin{pmatrix} B \\ C \end{pmatrix} \right) \circ \begin{pmatrix} C \\ D \end{pmatrix}$$

$$= \begin{pmatrix} A \\ C \end{pmatrix} \circ \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} A \\ D \end{pmatrix}$$

$$\text{and } F_o (goh) = \begin{pmatrix} A \\ B \end{pmatrix} \circ \left(\begin{pmatrix} B \\ C \end{pmatrix} \circ \begin{pmatrix} C \\ D \end{pmatrix} \right)$$

$$= \begin{pmatrix} A \\ B \end{pmatrix} \circ \begin{pmatrix} B \\ D \end{pmatrix} = \begin{pmatrix} A \\ D \end{pmatrix}$$

$$\text{Hence } (fog) oh = fo (goh)$$

Commutativity :

Permutation multiplication is not in general commutative.

Example :

Consider the two permutation viz.

$$F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}; \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

By rearranging g, we get

$$g = \begin{pmatrix} 3 & 4 & 2 & 1 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

$$\begin{aligned} \text{Then } fog &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 3 & 4 & 2 & 1 \\ 2 & 3 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{Again } gof &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 4 & 1 & 2 & 3 \\ 1 & 3 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \quad (\text{rearranging } f) \end{aligned}$$

Thus $fog \neq gof$.

Example :

Express each of the following permutations on $S = \{1, 2, 3, 4, 5\}$ as products of transpositions.

a. $(1 \ 3 \ 5)$, b. $(2 \ 3 \ 4 \ 5)$ c. $(1 \ 2 \ 3 \ 4 \ 5)$

Proof :

$$\begin{aligned} \text{(a) } (1 \ 3 \ 5) &= (1 \ 3) \circ (1 \ 5) \\ &= (1 \ 5) \circ (3 \ 5) \\ &= (1 \ 5) \circ (1 \ 3) \circ (1 \ 5) \circ (1 \ 3) \end{aligned}$$

$$\begin{aligned} \text{(b) } (2 \ 3 \ 4 \ 5) &= (2 \ 3) \circ (2 \ 4) \circ (2 \ 5) \\ &= (2 \ 5) \circ (3 \ 4) \circ (3 \ 5) \end{aligned}$$

$$\text{(c) } (1 \ 2 \ 3 \ 4 \ 5) = (1 \ 2) \circ (1 \ 3) \circ (1 \ 4) \circ (1 \ 5)$$

Note that the number of transpositions in (a) is even in (b) it is odd and in (c) it is even.

Example : Express

$$(a) (1\ 3\ 2\ 5) \circ (1\ 4\ 3) \circ (2\ 5\ 1)$$

(b) $(1\ 4\ 3\ 2) \circ (2\ 4\ 1) \circ (1\ 3\ 5)$ as the product of disjoint cycles.

Proof :

(a) under the first cycle, 1 goes to 3. Then under the second cycle 3 goes to 1. Finally under the third cycle 1 goes to 2. Then we have $(1\ 2\ \dots)$. Now under the first cycle 2 goes to 5. Then under the third cycle. 5 goes to 1. Thus we see that 1 goes to 2 and 2 goes to 1. Hence we have $(1\ 2)$ Again 3 goes to 2 under the first cycle. Then 2 goes to 5 under the third cycle. Thus 3 goes to 5. So that we have $(1\ 2)(3\ 5)$ Now 5 goes to 1. (first cycle) and 1 goes to 4 (second cycle). So that 5 goes to 4.

Hence finally we have $(1\ 2)(3\ 5\ 4)$ that is $(1\ 3\ 2\ 5) \circ (1\ 4\ 3) \circ (2\ 5\ 1) = (1\ 2) \circ (3\ 5\ 4)$.

(b) proceeding as in (a), we shall get

$$(1\ 4\ 3\ 2) \circ (2\ 4\ 1) \circ (1\ 3\ 5)$$

$$= (1\ 3\ 4\ 5) \circ (2) = (1\ 3\ 4\ 5)$$

General Law of Commutativity of the elements of a group :

Thm :

If in a group G .

a_1, a_2, \dots, a_n is any system of elements. Commutative in pairs and

$\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$ is any permutation of the set of n objects.

$1, 2, \dots, n$

$$\text{Then } a_1, a_2, \dots, a_n = a_{i_1} a_{i_2} \dots a_{i_n}$$

Solution :

We shall prove the result by induction method. Suppose the result is true for product of $n-1$ (or) less elements.

Then we shall show that it is also true for products of n elements. Two cases arise :

Case I:

Let $n=n$. In this case, we have

$$a_1 a_2 \dots a_n = (a_1 a_2 \dots a_{n-1}) a_n. (\because \text{composition in } G \text{ is associative})$$

$$= \left(a_{i_1} a_{i_2} \dots a_{i_{n-1}} \right) a_n$$

= \because according to supposition the result is true for $(n-1)$ elements ($\because n=in$)

$$= \left(a_{i_1} a_{i_2} \dots a_{i_{n-1}} \right) a_{i_n}$$

$$= a_{i_1} a_{i_2} \dots a_{i_{n-1}} a_{i_n} \text{ (by associativity)}$$

Case II :

$i_n = K$ when $K < n$ then

$$a_1 a_2 \dots a_{k-1} a_k a_{k+1} \dots a_n$$

$$= (a_1 \dots a_{k-1}) (a_k a_{k+1} \dots a_n) \text{ (by associativity)}$$

$$= (a_1 a_2 \dots a_{k-1}) (a_{k+1} a_n a_k) \text{ (by supposition)}$$

$$= (a_1 a_2 \dots a_{k-1}) \left[(a_{k+1} \dots a_n) a_k \right]$$

$$= \left[(a_1 a_2 \dots a_{k-1}) (a_{k+1} \dots a_n) \right] a_k$$

$$= \left[(a_1 a_2 \dots a_{k-1} a_{k+1} \dots a_n) \right] a_k$$

$$= \left(a_{i_1} a_{i_2} \dots a_{i_{n-1}} \right) a_{i_n}$$

$$= a_{i_1} a_{i_2} \dots a_{i_{n-1}} a_{i_n}$$

Now obviously the result is true for products of two elements.

Hence the proof is complete by induction.

Permutation groups :

Thm :

The set P_n of $n!$ permutations of n symbols forms a finite non abelian group under the operation of permutation multiplication.

Proof :

We have seen that the product of any two permutations is also a permutation on n symbols so that the closure axiom is satisfied. The other three group postulates are also satisfied as shown below.

P1 : The permutation multiplication is associative.

P2 : The identity permutation.

$I = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$ is the identity element.

P₃ : Every Permutation

$f = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$ has a permutation.

for its inverse defined by

$$f^{-1} = \begin{pmatrix} b_1 & b_2 & \dots & b_n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

It can be easily defined that

$$ff^{-1} = f^{-1}f = I.$$

Again since permutation multiplication is not in general commutative, the set P_n forms a finite non abelian group under the operation of permutation multiplication.

Thm :

The set A_n of $\frac{1}{2}n!$ even permutations forms a finite non abelian group under permutation multiplication.

Proof :

Since the product of two even permutations is an even permutation, the closure property is satisfied. The permutation multiplication has been shown to be associative.

The identity permutation I is the identity element. It can be shown that I is even.

Again we know that the product of two even permutations is an even permutation. Hence if f is an even permutation so also is f^{-1} . Since $ff^{-1} = I$. (an even permutation).

Hence every element of A_n has an inverse in A_n . Thus all the group postulates are satisfied. Since the permutation multiplication is not commutative. The set A_n forms a finite non abelian group. This is known as alternating group of order $\frac{1}{2}n!$.

2.3. Cyclic groups

Definitions :

A group G is called cyclic if, for some $a \in G$, every element $x \in G$ is of the form a^n , where n is some integer. The element a is then called a generator of G .

There may be more than one generators of a cyclic group. If G is a cyclic group generated by a . Then we shall write $G = \langle a \rangle$ (or) $G = (a)$. The elements of G will be of the form.

$a^{-3}, a^{-2}, a^{-1}, a^0 = e, a, a^2, \dots$ ofcourse they are not necessarily distinct.

Example : 1

The multiplicative group $G = \{1, -1, i, -i\}$ is cyclic.

We can write $G = \{i, i^2, i^3, i^4, \dots\}$ Thus G is a cyclic group and i is a generator. Also we can write,

$$G = \{-i, (-i)^2, (-i)^3, (-i)^4, \dots\}$$

Thus $(-i)$ is also a generator of G .

Example : 2

The multiplicative group $\{1, w, w^2\}$ is cyclic. The generators are w and w^2 .

Example : 3

Suppose G is any group and $a \in G$. Let H be the subgroup of G consisting of all integral powers of a .

(i.e.) $H = \{a^n / n \in I \text{ (the set of integers)}\}$

Then H is a cyclic subgroup of G generated by a .

Some properties of cyclic groups :

Thm :

Every cyclic group is an abelian group.

Proof :

Let $G = \langle a \rangle$ be a cyclic group generated by a . Let x, y be any two elements of G . Then there exists integers r and s such that

$$x = a^r; y = a^s \text{ now } xy = a^r a^s = a^{r+s} = a^{s+r}.$$

$$= a^s a^r = yx.$$

Thus we have $xy = yx \forall x, y \in G$.

$\therefore G$ is abelian.

Thm :

If 'a' is a generator of a cyclic group G . Then a^{-1} is also a generator of G .

Proof :

Let $G = \{a\}$ be a cyclic group generated by a . Let a^r be any element of G . Where r is some interger. We can write $a^r = (a^{-1})^{-r}$. Since $-r$ is also some integer therefore each element of G is generated by a^{-1} . Thus a^{-1} is also a generator of G .

Thm :

A cyclic group G with generation or of finite order n , is isomorphic to the multiplicative group of 'n' n^{th} roots of unity.

Proof :

Let 'a' be a generator of the cyclic group G . Since the order of 'a' is n , therefore 'n' is the least positive integer such that, $a^n = e$.

We shall show that the group G has exacty n distinct elements.

$$a, a^2, a^3 \dots a^n = e = a^0 \text{ ----- (1)}$$

No two elements of (1) can be equal. For if possible, Let $a^r = a^s$;

$$1 \leq s < r \leq n.$$

$$\text{Then } a^{r-s} = a^0 = e.$$

$$\text{Since } 0 < r-s < n \quad \therefore a^{r-s} = e \Rightarrow \text{the order of 'a' is less than } n.$$

$$\text{Hence } a^r \neq a^s.$$

\therefore All the n elements in (1) are distinct, Again Let a^t be any element of G . By division algorithm, there exists two integers P and q such that $t = np + q$; $0 \leq q < n$.

$$\begin{aligned} \therefore a^t &= a^{np+q} = a^{np} a^q = (a^n)^P a^q \\ &= e^P a^q = e \cdot a^q = a^q. \end{aligned}$$

Since $0 \leq q < n$ therefore a^q is one of the n elements in (1) thus each element of G is equal to some member of (1). There fore G has exactly n elements given in (1). Then $O(a) = O(G)$. We shall now show that G is isomorphic to the multiplicative group G' of the 'n' n^{th} roots of units namely.

$$1 = e^{\frac{2\pi i 0}{n}}, e^{\frac{2\pi i 1}{n}}, e^{\frac{4\pi i}{n}}, \dots, e^{\frac{2\pi i r}{n}}, \dots, e^{\frac{2\pi i (n-1)}{n}}$$

Consider the mapping $f : G \rightarrow G'$ defined by

$$f(a^r) = e^{\frac{2\pi i r}{n}} \quad \text{Where } 0 \leq r \leq n-1.$$

The mapping f is one-one.

Since $f(a^r) = f(a^s)$ where $0 \leq r \leq n-1$, $0 \leq s \leq n-1$.

$$\Rightarrow e^{\frac{2\pi i r}{n}} = e^{\frac{2\pi i s}{n}}, \Rightarrow r = s \Rightarrow a^r = a^s.$$

Again the number of elements in G is equal to the number of elements in G' . Therefore f is one-one implies f must be onto G' .

$$\text{Finally } f(a^r a^s) = f(a^{r+s}) = f(a^{nu+k})$$

Where u is some integer and $0 \leq k < n$.

We can write $(r+s) / n = u + k/n$.

$$\begin{aligned} \Rightarrow f(a^{nu+k}) &= \left[f(a^n)^u a^k \right] \\ &= f(a^k) \\ &= e^{2\pi i k/n} \\ &= e^{2\pi i nu/n} e^{2\pi i k/n} \end{aligned}$$

$$= \begin{pmatrix} \because a^n = e \\ e^{2\pi i u} = 1 \end{pmatrix}$$

$$\begin{aligned} &= e^{2\pi i (nu+k)/n} = e^{2\pi i (r+s)/n} \\ &= e^{2\pi i r/n} e^{2\pi i s/n} \\ &= f(a^r) f(a^s). \end{aligned}$$

Therefore f preserves compositions in G and G' . Hence G is isomorphic to G' .

Since every finite cyclic group of order n is isomorphic to the multiplicative group of ' n ' n th roots of unity. Therefore we can say that there is one and only one cyclic group of order n .

Thm :

A cyclic group G with a generator of finite order n , is isomorphic to the additive group of residue classes modulo n .

Proof :

First to prove that the group G has exactly n distinct elements, give the same proof as in *above theorem*. The group G' is here the group of residue classes modulo n .

For any integer ' a ' let (a) denote the residue class of the set of integers modulo n .

Then $G' = \{(a) \mid a \in I \text{ where } I \text{ is the set of integers}\}$. The group G' has only n distinct elements and we have

$$G' = \{(0) (1) (2) \dots (n-1)\}$$

$$\text{Also } G = \{a^r \mid r \in I\}$$

Consider the mapping $f: G \rightarrow G'$ defined by

$$f(a^r) = (r) \quad \forall r \in I.$$

First we must show that the mapping f is well defined.

Let $r, s \in I$ be such that $a^r = a^s$.

Then we must show that $f(a^r) = f(a^s)$

We have $a^r = a^s$.

$$\Rightarrow a^r a^{-s} = a^s a^{-s} \Rightarrow a^{r-s} = e$$

$$\Rightarrow n \text{ is a division of } r-s.$$

$$\Rightarrow r \equiv s \pmod{n} \Rightarrow (r) = (s)$$

$$\Rightarrow f(a^r) = f(a^s)$$

\therefore the mapping f is well defined.

f is one-one; Let a^r, a^s be any two elements of G where $r, s \in I$.

we have $f(a^r) = f(a^s)$

$$\Rightarrow (r) = (s)$$

$$\Rightarrow r - s \text{ is divisible by } n.$$

$$\Rightarrow r - s = Kn \text{ where } k \in I.$$

$$\Rightarrow a^{r-s} = a^{kn} \Rightarrow a^r a^{-s} = (a^n)^k.$$

$$\Rightarrow a^r a^{-s} = e^k \Rightarrow a^r a^{-s} = e.$$

$$\Rightarrow a^r = a^s.$$

$$\Rightarrow f \text{ is one-one.}$$

f is on to let (r) be any element of G' . Then r is an integer.

We have $a^r \in G$ and $f(a^r) = (r)$. Therefore f is onto G' .

f preserves Compositions.

$$\text{We have } f(a^r a^s) = f(a^{r+s}) = (r+s)$$

$$= (r) + (s)$$

$$= f(a^r) + f(a^s)$$

$$\therefore G \cong G'$$

Thm :

If a finite group of order n contains an element of order n . The group must be cyclic.

Proof :

Suppose G is a finite group, of order n , Let $a \in G$ and let n be the order of a . If H is the cyclic subgroup of G generated by a (i.e.) if $H = \{a^r / r \in I\}$ then the order of H is n because the order of the generator ' a ' of H is n . Thus H is a cyclic subgroup of G and the order of H is equal to the order of G . Hence $H=G$ and therefore G itself is a cyclic group and ' a ' is generator of G .

Thm :

Every group of prime order is cyclic.

Proof :

Suppose G is a finite group whose order is a prime number P_1 then to prove that G is a cyclic group. Note that an integer P is said to be a prime number if $P \neq 0$, $P \neq \pm 1$ and if the only divisors of P are $\pm 1, \pm P$.

Since G is a group of prime order, therefore G must contain atleast 2 elements. Note that 2 is the least positive prime integer. Therefore there must exist an element $a \in G$ such that $a \neq$ the identity element e .

Since ' a ' is not the identity element therefore $o(a)$ is definitely ≥ 2 . Let $o(a) = m$. If H is the cyclic subgroup of G generated by ' a '. Then $O(H) = O(a) = m$. By Lagrange's theorem ' m ' must be a divisor of P . But P is prime and $m \geq 2$. Hence $m=P$.

$\therefore H=G$ since H is cyclic. Therefore G is cyclic and a is a generator of G .

Note :

Every group of prime order P is cyclic. Also every cyclic group of order P is isomorphic to the additive group of residue classes module P . Therefore we can say that if P is prime then there one and only one group of order P .

Thm :

If a cyclic group G is generated by an element ' a ' of order n , then a^m is a generator of $G \Leftrightarrow$ the greatest common divisor of m and n is 1 (ie) iff m and n are relative primes.

Proof :

Suppose m is relatively prime to n . Consider the cyclic subgroup $H = \{a^m\}$ of G generated by a^m obviously $H \leq G$. Since each integral power of a^m will also be an integral power of a .

Since m is relatively prime to n , therefore there exist two integers u, v such that $um + vn = 1$

$$\therefore a^{um+vn} = a^1.$$

$$\Rightarrow a^{um} a^{vn} = a^1.$$

$$\Rightarrow (a^m)^u = a \text{ since } a^{vn} = (a^n)^v = e^v = e.$$

\therefore each integral power of ' a ' will also be some integral power of a^m . Therefore $G \leq H$. Hence $H = G$ and a^m is a generator of G .

Converse :

Suppose a^m is a generator of G . Let the greatest common divisor of m and n be d and $d \neq 1$ (i.e) $d > 1$. Then m/d and n/d must be integers.

$$\text{Now } (a^m)^{n/d} = (a^n)^{m/d} = e^{m/d} = e$$

Obviously n/d is a positive integer less than n itself. Thus $O(a^m) < n$. Therefore a^m cannot be a generator of G because the order of a^m is not equal to the order of G . Hence d must be equal to 1. Thus m is prime to n .

Note :

If G is a cyclic group of order n . Then the total number of generators of G will be equal to the number of integers less than n and prime to n . For example if a is a generator of a cyclic group G of order 8, then a^3, a^5, a^7 will be the only other generators of G . Since 4 is not prime to G . Similarly a^2, a^6, a^8 cannot be generators of G .

If G is a cyclic group of prime order P generated by ' a '. Then a, a^2, \dots, a^{P-1} are all generators of G and thus G has $P-1$ generators.

Thm :-

If G is an infinite cyclic group. Then G has exactly two generators and G is isomorphic to the additive group of integers.

Proof:-

Let $G = \{a\}$ be an infinite cyclic group generated by a . The elements of G will be integral powers of a . We claim that no two distinct integral powers of ' a ' can be equal.

For if possible.

$$\text{Let } a^r = a^s ; r > s$$

$$\text{Then } a^r a^{-s} = a^s a^{-s} = a^0 = e$$

$$\therefore a^{r-s} = e.$$

Since $r-s$ is a positive integer, therefore $a^{r-s} = e$ implies that $O(a)$ is finite. So ' a ' cannot be generator of an infinite cyclic group G . Hence $a^r \neq a^s$. Unless $r = s$.

$$\therefore \text{ We can write } G = \{ \dots a^{-3}, a^{-2}, a^{-1}, a^0 = e, a, a^2, \dots \}$$

If a^r is any element of G . We can write $a^r = (a^{-1})^{-r}$. Thus a^{-1} is also a generator of G . Also as proved above a and a^{-1} are distinct element of G .

Now if $m \neq 1$ (or) -1 then a^m cannot be a generator of G . If a^m is to be a generator of G . There must exist an integer k such that $(a^m)^k = a$ (i.e.) $a^{mk} = a$. Now $m \neq 1$ (or) $-1 \Rightarrow m^k \neq 1$. Therefore two distinct integral powers of ' a ' are equal and this contradicts the statement we have just proved. Hence a^m cannot be a generator of G if $m \neq 1$ (or) -1 . Thus G has exactly two generator

(i.e.) a and a^{-1}

Let I be the additive group of integers.

$$(i.e.) I = \{ \dots -3, -2, -1, 0, 1, 2, 3, \dots \}$$

To prove that $G \cong I$.

Consider the mapping $\phi: G \rightarrow I$ defined by $\phi(a^m) = m \forall m \in I$.

The mapping ϕ is one - one because

$$\phi(a^m) = \phi(a^n) \Rightarrow m = n \Rightarrow a^m = a^n$$

Obviously ϕ is onto I .

Also $\phi(a^m a^n) = \phi(a^{m+n}) = m+n \phi(a) = m \phi(a) + n \phi(a)$

Hence ϕ is an isomorphism of G onto I .

Note :

Since every infinite cyclic group is isomorphic to the additive group of integers, therefore we can say that there is one and only one infinite cyclic group.

Thm :-

Every subgroup of cyclic group is cyclic.

Proof :-

Suppose $G = \{a\}$ is a cyclic group generated by a . If $H = G$ (or) $\{e\}$ then obviously H is cyclic. So Let H be a proper subgroup of G . The elements of H are integral power of a . If $a^s \in H$ then the inverse of (i.e.) $a^{-s} \in H$. $\therefore H$ contains elements which are positive as well as negative integral powers of a . Let 'm' be the least positive integer such that $a^m \in H$. Then we shall prove that $H = \{a^m\}$.

(i.e.) H is cyclic and is generated by a^m .

Let a^t be any arbitrary element of H . By division algorithm there exists integers q and r such that $t = mq + r$; $0 \leq r < m$.

Now $a^m \in H \Rightarrow (a^m)^q \in H$ (by closure property)

$$\Rightarrow a^{mq} \in H \Rightarrow (a^{mq})^{-1} \in H$$

$$\Rightarrow a^{-mq} \in H.$$

Also $a^t \in H$, $a^{-mq} \in H$

$$\Rightarrow a^t a^{-mq} \in H \Rightarrow a^{t-mq} \in H$$

$$\Rightarrow a^r \in H \quad (\because r = t - mq).$$

Now m is the least positive integer such that $a^m \in H$ and $0 \leq r < m$.

$\therefore r$ must be equal to zero. hence $t = mq$

$$\therefore a^t = a^{mq} = (a^m)^q$$

Thus every element $a^t \in H$ is of the form $(a^m)^q$. Therefore H is cyclic and a^m is a generator of H .

Example : 1

Show that the group $(\{1, 2, 3, 4, 5, 6\} \times 7)$ is cyclic. How many generators are there?

Let us denote the given group by G .

If there exists an element $a \in G$ the group G . Then the group G will be a cyclic group and 'a' will be a generator of G .

We see that $O(3) = 6$ because

$$3^1 = 3, 3^2 = 3 \times 7^3 = 2$$

$$\begin{aligned} 3^3 &= 3^2 \times 7^3 \\ &= 2 \times 7^3 \\ &= 6 \end{aligned}$$

$$\begin{aligned} 3^4 &= 6 \times 7^3 \\ &= 4 \end{aligned}$$

$$3^5 = 3^4 \times 7^3$$

$$= 4 \times 7^3$$

$$= 5$$

$$3^6 = 5 \times 7^3$$

$$= 1 \text{ (i.e.) identity element.}$$

G is cyclic and 3 is a generator of G . We can write.

$$G = \{3, 3^2, 3^3, 3^4, 3^5, 3^6\}$$

Now 5 is prime to 6. Therefore 3^5 .

(i.e.) 5 is also a generator of G .

UNIT - 3

3.1 Cosets and Lagrange's theorem

Consider S_3 ; Let $H = \{e, P_3\}$. H is a subgroup of S_3 . This subgroup does not contain the elements P_1, P_2, P_4 and P_5 . Let us now perform the binary operation between P_1 and each element of H . We denote the resultant set by the symbol $P_1 H$. Thus $P_1 H = \{P_1 e, P_1 P_3\} = \{P_1, P_4\}$

Now the element P_2 belongs neither to H nor to $P_1 H$ therefore we now perform the binary operation between P_2 and the elements of H . Thus $P_2 H = \{P_2 e, P_2 P_3\} = \{P_2, P_5\}$. The union of the three sets $H, P_1 H, P_2 H$ gives all the elements of S_3 .

$$(i.e) S_3 = H \cup P_1 H \cup P_2 H.$$

Further $H, P_1 H$ and $P_2 H$ are mutually disjoint. Hence $\{H, P_1 H, P_2 H\}$ is a partition of S_3 .

Definition :

Let H be a subgroup of a group G . Let $a \in G$ then the set $aH = \{ah/h \in H\}$ is called the left coset of H defined by a in G . Similarly $Ha = \{ha/h \in H\}$ is called the right coset of H defined by ' a '.

Examples :

1) Let us determine the left cosets of $5\mathbb{Z}, +$ in $(\mathbb{Z}, +)$. Hence the operation is addition. $0 + 5\mathbb{Z} = 5\mathbb{Z}$ is itself is a left coset. Another left coset is $1 + 5\mathbb{Z} = \{1 + 5n/n \in \mathbb{Z}\}$. We notice that this left coset contains all integers having remainder 1. When divided by 5.

$$\text{Similarly } 2 + 5\mathbb{Z} = \{2 + 5n / n \in \mathbb{Z}\}$$

$$3 + 5\mathbb{Z} = \{3 + 5n / n \in \mathbb{Z}\}$$

$$\text{and } 4 + 5\mathbb{Z} = \{4 + 5n / n \in \mathbb{Z}\}$$

These are all the left cosets of $(5\mathbb{Z}, +)$. Here also we note that all the left cosets are mutually disjoint and their union is \mathbb{Z} . In other words the collection of all left cosets form a partition of the group.

2) Consider $(\mathbb{Z}_{12}, \oplus)$ $H = \{0, 4, 8\}$ is a subgroup of G . The left cosets of H are given by

$$0 + H = \{0, 4, 8\} = H$$

$$1 + H = \{1, 5, 9\}$$

$$2 + H = \{2, 6, 10\}$$

$$\text{and } 3 + H = \{3, 7, 11\}$$

we notice that $4 + H = \{4, 8, 0\} = H$

and $5 + H = \{5, 9, 1\} = 1 + H$

Exercise :

1) Find all the left cosets of $(\mathbb{Z}, +) : n(\mathbb{Z}, +)$

Thm : 3.1

Let G be a group and H be a subgroup of G .

Then (i) $a \in H \Leftrightarrow aH = H$

(ii) $aH = bH \Leftrightarrow a^{-1}b \in H$

(iii) $a \in bH \Leftrightarrow a^{-1} \in Hb^{-1}$

(iv) $a \in bH \Leftrightarrow aH = bH$

Pf :

(i) Let $a \in H$ we claim that $aH = H$. Let $x \in H$ then $x = ah$ for some $h \in H$. Now $a \in H$ and $h \in H \Rightarrow ah = x \in H$. Since H is a subgroup.

Hence $aH \subseteq H$

Let $x \in H$ Then $x = a(a^{-1}x) \in aH$.

Hence $H \subseteq aH$ Thus $H = aH$

Conversely Let $aH = H$. Now $a = ae \in aH$.

$\therefore a \in H$.

(ii) Let $aH = bH$

$\therefore a^{-1}(aH) = a^{-1}(bH)$

$\therefore H = (a^{-1}b)H$

$\therefore ab \in H$ by (i)

conversely Let $a^{-1}b \in H$

Then $a^{-1}bH = H$ by (i)

$\therefore aa^{-1}bH = aH$ and hence $bH = aH$

(iii) Let $a \in bH$ then $a = bh$ for some $h \in H$.

$\therefore a^{-1} = (bh)^{-1} = b^{-1}h^{-1} \in b^{-1}H$

Converse can be similarly proved.

(iv) Let $a \in bH$ we claim that $aH = bH$

Let $x \in aH$ Then $x = ah_1$, for some $h_1 \in H$

Also $a \in bH \Rightarrow a = bh_2$ for some $h_2 \in H$ (1)

$$\therefore x = (bh_2)h_1 = b(h_2h_1) \in bH$$

$$aH \subseteq bH.$$

Now let $x \in bH$ Then $x = bh_3$ for some $h_3 \in H$. Also from (1) $b = ah_2^{-1}$

$$\therefore x = ah_2^{-1}h_3 \in aH.$$

$$\therefore bH \subseteq aH \quad \text{Hence } aH = bH$$

Conversely Let $aH = bH$ Then $a = ac \in aH$

$$\therefore a \in bH.$$

Theorem : 3.2

Let H be a subgroup of G . Then

- (i) any two left cosets of H are either identical (or) disjoint
- (ii) Union of all the left cosets of H is G .
- (iii) the number of elements in any left coset aH is the same as the number of elements in H .

Pf : (i) Let aH and bH be two left cosets suppose aH and bH are not disjoint we claim that $aH = bH$.

Since aH and bH are not disjoint $aH \cap bH \neq \emptyset$

\therefore There exists an element $c \in aH \cap bH$.

$$\therefore c \in aH \quad \text{and } c \in bH$$

$\therefore aH = cH$ and $bH = cH$ by (iv) of Theorem (1)

$$\therefore aH = bH$$

(ii) Let $a \in G$. Then $a = ae \in aH$

\therefore Every element of G belongs to a left coset of H .

\therefore The union of all the left cosets of H is G .

H is G .

(iii) The map $f: H \rightarrow aH$ defined by $f(h) = ah$ is clearly a bijection.

Hence every left coset has the same number of elements as H .

Note - 1 : This theorem shows that the collection of all left cosets forms a partition of the group.

Note - 2 : The above result is true if we replace left cosets by right cosets in what follows the results we prove for left cosets are also true for right cosets.

Remark :

Let H be a subgroup of G . We define a relation in G as follows. Define $a \sim b \Leftrightarrow a^{-1}b \in H$. Then \sim is an equivalence relation.

(or) $a^{-1}a = e \in H$. Hence $a \sim a$. Hence \sim is reflexive.

$a \sim b \Leftrightarrow a^{-1}b \in H \Rightarrow (a^{-1}b)^{-1} \in H$.

$\Rightarrow b^{-1}a \in H \Rightarrow b \sim a$

$\therefore a \sim b \Rightarrow b \sim a$. Hence \sim is symmetric.

Now $a \sim b$ and $b \sim c \Rightarrow a^{-1}b \in H$ and $b^{-1}c \in H$

$\Rightarrow (a^{-1}b)(b^{-1}c) \in H$

$\Rightarrow a^{-1}c \in H$

$\Rightarrow a \sim c$

Hence \sim is transitive.

Thus \sim is an equivalence relation. Now we claim that equivalence class $[a] = aH$

Let $b \in [a]$ Then $b \sim a$

$\therefore a^{-1}b \in H$.

$a^{-1}b = h$ for some $h \in H$

$\therefore b = ah$ hence $b \in aH$

$\therefore [a] \subseteq aH$

Also $b \in aH \Rightarrow b = ah$ for some $h \in H$

$\Rightarrow a^{-1}b = h \in H$

$\Rightarrow a \sim b$

$\Rightarrow b \in [a]$

$aH \subseteq [a]$ Hence $[a] = aH$

Thus the left cosets of H in G are precisely the equivalence classes determined by \sim . Hence the left cosets form a partition of G . This gives another proof of thm 3.2.

Thm 3.3 :

Let G be a group and H be a subgroup of G . The number of Cosets of H is the same as the number of right cosets of H .

Pf :

Let L and R respectively denote the set of left and right cosets of H . We define a map $f: L \rightarrow R$ by $f(aH) = Ha$

f is well defined. For $aH = bH$.

$$\Rightarrow a^{-1}b \in H$$

$$\Rightarrow a^{-1} \in Hb^{-1}$$

$$\Rightarrow Ha^{-1} = Hb^{-1}$$

f is 1-1 for $f(aH) = f(bH)$

$$\Rightarrow Ha^{-1} = Hb^{-1}$$

$$\Rightarrow a^{-1} \in Hb^{-1}$$

$$\Rightarrow a^{-1} = Hb^{-1} \text{ for some } h \in H.$$

$$\Rightarrow a = hb^{-1}$$

$$\Rightarrow a \in Hb$$

$$\Rightarrow aH = bH$$

f is onto for every right coset Ha has a pre image under f namely $a^{-1}H$. Hence f is a bijection from L to R . Hence the number of left cosets is the same as the number of right cosets.

Definition :

Let H be a subgroup of G . The number of distinct left (right) cosets of H is called the index of H in G and is denoted by $[G:H]$

Examples :

In (\mathbb{Z}_8, \oplus) $H = \{0, 4\}$ is a subgroup. The left cosets of H are given by

$$0 + H = \{0, 4\} = H$$

$$1 + H = \{1, 5\}$$

$$2 + H = \{2, 6\}$$

$$3 + H = \{3, 7\}$$

These are four distinct left cosets of H . Hence the index of the subgroup H is 4. Note that $\{Z_8:H\} \times |H| = 4 \times 2 = 8 = |Z_8|$

Exercises :

- 1) Find the index of $(nz, +)$ in $(z, +)$
- 2) Find the index of $(8z, +)$ in $(2z, +)$

Lagrange's theorem :

Let G be a finite group of order n and H be any subgroup of G . Then the order of H divides the order of G .

Pf : Let $|H| = m$ and $[G:H] = r$

Then the number of distinct left cosets of H in G is r .

By the thm 3.1 these r left cosets are mutually disjoint they have the same number of elements namely m and their union is G .

$$\therefore n = rm \quad \text{Hence } m \text{ divides } n.$$

Corollary :

$$[G : H] = \frac{|G|}{|H|}$$

Note : 1

Lagrange's theorem has many important applications in group theory. For example a group G of order 8 cannot have subgroups of order 3, 5, 6 (or) 7. In fact any proper subgroup of G must be of order 2 or 4.

Note : 2

Any group of prime order has no proper subgroups.

Note : 3

The converse of Lagrange's thm is false (ie) If G is a group of order n and m divides n , then G need not have a subgroup of order m for example A_4 a group of order 12 does not have a subgroup of order 6.

However there are groups in which the converse of Lagrange's theorem is true. For example consider S_3 . This is a group of order 6. $\{e, P_4\}$ is a subgroup of order 2 and $\{e, P_1, P_2\}$ is a subgroup of order 3. Hence for every division m of 6 there is a subgroup of S_3 of order m .

Exercises :

- 1) Show that the converse of Lagrange's theorem is true in any finite cyclic group.

Theorem 3.4 :

The order of any element of a finite group G divides the order of G .

Pf : Let G a group of order n . Let $a \in G$ be an element of order m . Then the order of ' a ' is the same as the order of the cyclic group $\langle a \rangle$. Now by Lagrange's theorem the order of the subgroup $\langle a \rangle$ divides the order of G . Hence m/n .

Theorem 3.5 :

Every group of prime order is cyclic.

Pf : Let G be a group of order P . Where P is prime. Let $a \in G$ and $a \neq e$.

By the theorem 3.4

order of ' a ' divides P .

Since $a \neq e$ order of a is P .

Hence $G = \langle a \rangle$ So that G is cyclic.

Theorem 3.6 :

Let G be a group of order n . Let $a \in G$ then $a^n = e$.

Pf : Let the order of a be m . Then m divides n . Hence $n = mq$.

$$\therefore a^n = a^{mq} = (a^m)^q = e^q = e.$$

3.2 Euler's Theorem and Fermat's theorem

Theorem 3.7 : Euler's Theorem

If n is any integer and $(a, n) = 1$ then $a^{\phi(n)} \equiv 1 \pmod{n}$.

Pf : Let $G = \{m/m < n \text{ and } (m, n) = 1\}$ G is a group under multiplication modulo n . This group is of order $\phi(n)$

Now let $(a, n) = 1$

Let $a = qn + r : 0 \leq r < n$ so that $a \equiv r \pmod{n}$

Since $(a, n) = 1$ we have $(n, r) = 1$ so that $r \in G$.

$$\therefore r^{\phi(n)} \equiv 1 \pmod{n} \text{ (by thm 3.6)}$$

$$r^{\phi(n)} \equiv 1 \pmod{n}$$

$$\text{Also } a^{\phi(n)} \equiv r^{\phi(n)} \pmod{n}.$$

So that $a^{\phi(n)} \equiv 1 \pmod{n}$. (Since \equiv is transitive)

Fermat's theorem :

Let P be a prime number and a be any integer relatively prime to P . Then $a^{P-1} \equiv 1 \pmod{P}$.

Pf : Since P is prime, $\phi(p) = p - 1$ and hence the result follows from Euler's theorem.

Theorem 3.8 :

A group G has no proper subgroups iff it is a cyclic group of prime order.

Pf : Suppose G is a group of prime order. Then it follows from Lagrange's theorem that G has no proper subgroups.

Conversely Let G be a group having no proper subgroups. First we shall prove that G is cyclic.

Suppose G is not cyclic. Let $a \in G$ and $a \neq e$. Then the cyclic group $\langle a \rangle$ is a proper subgroup of G which is a contradiction.

Hence G is cyclic.

Also G cannot be infinite for an infinite cyclic group contains a proper subgroup $\langle a^2 \rangle$. Hence G must be of finite order say n . We claim that n is prime if possible.

Let n be a composite number.

Let $n = pq$ where $p, q > 1$

Let $a \in G$ be a generator of the group.

Then $\langle a^p \rangle$ is a subgroup of order q and hence is a proper subgroup of G which is a contradiction.

Hence n is prime.

$\therefore G$ is a cyclic group of prime order.

Solved problems :

1) Let A and B be subgroups of a finite group G such that A is a subgroup of B . Show that $[G:A] = [G:B] : [B:A]$

Solution :

$$[G : A] = \frac{|G|}{|A|} \quad \text{by Lagrange's thm.}$$

$$[G : B] = \frac{|G|}{|B|}$$

$$\text{and } [B : A] = \frac{|B|}{|A|}$$

$$\therefore [G:B] [B:A] = \frac{|G|}{|B|} \cdot \frac{|B|}{|A|} = \frac{|G|}{|A|} = [G : A]$$

2) Let A and B be two finite subgroups of a group G such that $|A|$ and $|B|$ have no common divisors. Then show that $A \cap B = \{e\}$

Pf : $A \cap B$ is a subgroup of A and B .

\therefore By Lagrange's theorem. $|A \cap B|$ divides $|A|$ and $|B|$

But by hypothesis $|A|$ and $|B|$ have no common divisors.

$\therefore |A \cap B| = 1$ Hence $A \cap B = \{e\}$

3) Let H and K be two subgroups of G of finite index in G . Prove that $H \cap K$ is a subgroup of finite index in G .

Pf : We know that $H \cap K$ is a subgroup of G . Let $[G:H] = m$ and $[G:K] = n$ we claim that for any $a \in G$.

$$(H \cap K) a = Ha \cap Ka$$

Clearly $H \cap K \subseteq H$ and K .

$$(H \cap K) a \subseteq Ha \text{ and } Ka.$$

$$\therefore (H \cap K) a \subseteq Ha \cap Ka \dots\dots\dots(1)$$

Now let $x \in Ha \cap Ka$

$$\therefore x \in Ha \text{ and } x \in Ka$$

$$\therefore x = ha \text{ for some } h \in H$$

$$x = ka \text{ for some } k \in K$$

$$\therefore x = ha = ka$$

$$\therefore h = k.$$

$$x \in H \cap K.$$

$$h \in (H \cap K) a$$

$$\therefore Ha \cap Ka \subseteq (H \cap K) a \dots\dots\dots(2)$$

from 1 and 2

$$(H \cap K) a = Ha \cap Ka.$$

\therefore Every right coset of $H \cap K$ in G is the intersection of a right coset of H and a right coset of K .

Also since $[G:H] = m$, the number of right cosets of H in G is m .

Similarly the number of right cosets of K in G is n .

Hence the number of right cosets of $H \cap K$ in G is at most mn .

Thm : Let $L = H \cap K$ be a subgroup of index 2 in a group G then H is a normal subgroup of G .

Pf : Let $L = H \cap K$. Since H and K are subgroups of G , L is also a subgroup of G and $L \subseteq H$ and K .

4) Let H and K be two finite subgroups of a group G then $|Hk| = \frac{|H| |K|}{|H \cap K|}$

Pf : Let $L = H \cap K$. Since H and K are subgroups of G , L is also a subgroup of G and $L \subseteq H$ and K .

Now that Lx_1, Lx_2, \dots, Lx_m be the distinct right cosets of L in K so that

$$K = Lx_1 \cup Lx_2 \cup \dots \cup Lx_m \quad \dots \dots \dots (1)$$

$$\text{and } m = [K:L] = \frac{|K|}{|L|} = \frac{|K|}{|H \cap K|} \quad \dots \dots \dots (2)$$

$$\begin{aligned} \text{From (1) } Hk &= HLx_1 \cup HLx_2 \cup \dots \cup HLx_m \\ &= Hx_1 \cup \dots \cup Hx_m \quad \text{Since } L \subseteq H \quad \dots \dots \dots (3) \end{aligned}$$

We claim that the cosets Hx_1, Hx_2, \dots, Hx_m are distinct.

Suppose $Hx_1 = Hx_2$

$$\therefore x_1 x_2^{-1} \in H$$

Also $x_1, x_2 \in K$ and hence $x_1, x_2^{-1} \in K$

$$\therefore x_1 x_2^{-1} \in H \cap K = L$$

Hence $Lx_1 = Lx_2$ which is a contradiction. Since the cosets Lx_1, Lx_2, \dots, Lx_m are distinct.

Hence from (3) we have

$$\begin{aligned} |HK| &= |Hx_1| + \dots + |Hx_m| \\ &= m |H| \\ &= \frac{|H| |K|}{|H \cap K|} \quad \text{by (2)} \end{aligned}$$

5) Let H and K be two subgroups of a finite group G such that $|H| > \sqrt{|G|}$ and $|K| > \sqrt{|G|}$

Then $H \cap K \neq \{e\}$

Pf : Suppose $H \cap K = \{e\}$

$$\therefore |H \cap K| = 1$$

$$\therefore |HK| = \frac{|H| |K|}{|H \cap K|}$$

$$= \frac{|H| |K|}{|H \cap K|}$$

$$> \sqrt{|G|} \sqrt{|G|} = |G|$$

$\therefore |HK| > |G|$ which is a contradiction.

$$\therefore |H \cap K| \neq \{e\}$$

3.3 Normal subgroups and quotient groups

Consider the subgroup $H = \{e, P_3\}$ of S_3 . Then $HP_1 = \{P_1, P_5\}$ and $P_1H = \{P_1, P_4\}$. Hence $HP_1 \neq P_1H$. Thus a left coset need not be equal to the corresponding right coset. However there are subgroups for which every left coset is same as the corresponding right coset.

For example Consider the subgroup $H = \{e, P_1, P_2\}$ of S_3 .

Clearly $Ha = aH = H$ for all $a \in H$ and $aH = Ha = S_3 - H = \{P_3, P_4, P_5\}$ for all $a \in H$.

$$\therefore aH = Ha \text{ for all } a \in S_3$$

Thus for some subgroups every left coset is a right coset. This leads to the following definition of a special class of subgroups.

Definition :

A subgroup H of G is called a normal subgroup of G if $aH = Ha$

Examples :

1) For any group G , $\{e\}$ and G are normal subgroups.

Thm : Every subgroup of an abelian group is a normal subgroup.

Pf : Let G be an abelian group and Let H be a subgroup of G . Let $a \in G$.

We claim that $aH = Ha$

Let $x \in aH$ then $x = ah$ for some $h \in H$

$$= ha \text{ (Since } G \text{ is abelian)}$$

$$\therefore x \in Ha \text{ Hence } aH \subseteq Ha$$

Similarly $Ha \subseteq aH$

$$\therefore aH = Ha \text{ and hence } H \text{ is a normal subgroup of } G.$$

Thm : Let H be a subgroup of index 2 in a group G then H is a normal subgroup of G .

Pf : If $a \in H$ then $H = aH = Ha$. If $a \notin H$ then aH is a left coset different from H .

$$\text{Hence } H \cap aH = \phi$$

Further since index of H in G is 2, $H \cup aH = G$ Hence $aH = G - H$

Similarly $Ha = G - H$ so that $aH = Ha$

Hence H is a normal subgroup of G .

Example :

The alternating group A_n is a subgroup of index 2 in S_n and hence is a normal subgroup of S_n .

Thm :

Let N be a subgroup of G . Then the following are equivalent

- (i) N is a normal subgroup of G .
- (ii) $aNa^{-1} = N$ for all $a \in G$
- (iii) $aNa^{-1} \subseteq N$ for all $a \in G$
- (iv) $ana^{-1} \in N$ for all $n \in N$ and $a \in G$.

Pf : (i) \Rightarrow (ii)

Suppose N is a normal subgroup of G .

$$\therefore aN = Na \text{ for all } a \in G$$

$$\therefore aNa^{-1} = Naa^{-1} = Ne = N$$

(ii) \Rightarrow (iii) and (iii) \Rightarrow (iv) are obvious.

(iv) \Rightarrow (i)

Suppose that $ana^{-1} \in N$ for all $n \in N$ and $a \in G$.

We claim that $aN = Na$

Let $x \in aN$

$$\therefore x = an \text{ for some } n \in N$$

$$\therefore x = (ana^{-1})a \in Na \text{ since } ana^{-1} \in N$$

$$\therefore aN \subseteq Na \dots\dots\dots(1)$$

Now let $x \in Na$

$$\therefore x = na \text{ for some } n \in N$$

$$x = a(a^{-1}na) = a(a^{-1}n(a^{-1})) \in aN$$

$$Na \subseteq aN \dots\dots\dots(2)$$

from (1) and (2) we get $Na = aN$. Hence N is a normal subgroup of G .

Problems :

1) Prove that the intersection of two normal subgroups of a group G is a normal subgroup of G .

Solution :

Let H and K be two normal subgroups of G then $H \cap K$ is a subgroup of G . Now let $a \in G$ and $x \in H \cap K$ then $x \in H$ and $x \in K$.

Since H and K are normal $axa^{-1} \in H$ and $axa^{-1} \in K$.

Hence $axa^{-1} \in H \cap K$. Thus $H \cap K$ is a normal subgroup of G .

2) The centre H of a group G is a normal subgroup of G .

Pf : The centre H of G is given by $H = \{a/a \in G : ax = xa \text{ for all } x \in G\}$.

Now let $x \in H$ and $a \in G$.

Hence $ax = xa$.

$$\therefore x = axa^{-1} \in H.$$

Hence H is a normal subgroup of G .

3) Let H be a subgroup of G . Let $a \in G$. Then aHa^{-1} is a subgroup of G .

Pf : $e = aea^{-1} \in aHa^{-1}$ and hence $aHa^{-1} \neq \phi$

Let $x, y \in aHa^{-1}$. Then $x = ah, a^{-1}$.

$$h_1, h_2 \in H$$

$$\begin{aligned} \text{Now } xy^{-1} &= (ah_1a^{-1})(ah_2a^{-1})^{-1} \\ &= (ah_1a^{-1})(ah_2^{-1}a^{-1}) \\ &= a(h_1h_2^{-1})a^{-1} \in aHa^{-1} \end{aligned}$$

$\therefore aHa^{-1}$ is a subgroup of G .

4) Show that if a group G has exactly one subgroup H of given order then H is a normal subgroup of G .

Pf : Let the order of H be m .

Let $a \in G$. Then by the above problem.

aHa^{-1} is also subgroup of G .

We claim that $|H| = |aHa^{-1}| = m$

Now consider $f : H \rightarrow aHa^{-1}$ defined by $f(h) = aha^{-1}$

f is 1-1 for $f(h_1) = f(h_2) \Rightarrow ah_1a^{-1} = ah_2a^{-1} \Rightarrow h_1 = h_2$

f is onto for Let $x = aha^{-1} \in aHa^{-1}$

Then $f(h) = x$ thus f is a bijection.

$\therefore |H| = |aHa^{-1}| = m$.

But H is the only subgroup of G of order m .

$\therefore aHa^{-1} = H$ Hence $aH = Ha$

H is a normal subgroup of G .

5) Show that if H and N are subgroups of a group G and N is normal in G . Then $H \cap N$ is normal in H . Show by an example that $H \cap N$ need not be normal in G .

Pf : Let $x \in H \cap N$ and $a \in H$.

Now claim that $axa^{-1} \in H \cap N$.

Now $x \in N$ and $a \in H \Rightarrow axa^{-1} \in N$. Since N is a normal subgroup.

Also $x \in H$ and $a \in H \Rightarrow axa^{-1} \in H$ Since H is a group. Hence $axa^{-1} \in H \cap N$.

$\therefore H \cap N$ is a normal subgroup of H .

The following example shows that $H \cap N$ need not be normal in G .

Let $G = S_3$ take $N = G$ and $H = \{e, P_3\}$

Now $H \cap N = H$ which is not normal in G .

6) If H is a subgroup of G and N is a normal subgroup of G then HN is a subgroup of G .

Pf : To prove that HN is a subgroup of G .

It is enough if we prove that $HN = NH$.

Let $x \in HN$. Then $x = hn$ where $h \in H$ and $n \in N$.

$\therefore x \in HN$.

But $hN = Nh$ since N is normal.

$x \in NH$ Hence $x = nh$ where $n \in N$

$\therefore x \in Nh$ hence $HN \subseteq NH$.

Similarly $NH \subseteq HN$.

$\therefore HN = NH$ Hence HN is a subgroup of G .

7) M and N are normal subgroups of a group G such that $M \cap N = \{e\}$. Show that every element of M commutes with every element of N .

Pf : Let $a \in N$ and $b \in N$

we claim that $ab = ba$

Consider the element $aba^{-1}b^{-1}$

Since $a^{-1} \in M$ and M is normal. $ba^{-1}b^{-1} \in M$.

Also $a \in M$ so that $aba^{-1}b^{-1} \in M$.

Again since $b \in N$ and N is normal $ab a^{-1} \in N$.

Also $b^{-1} \in N$. So that $aba^{-1}b^{-1} \in N$.

Thus $aba^{-1}b^{-1} \in M \cap N = \{e\}$.

$\therefore aba^{-1}b^{-1} = e$ so that $ab = ba$.

Thm : A subgroup N of G is normal \Leftrightarrow the product of two right cosets of N is again a right coset of N .

Pf : Suppose N is a normal subgroup of G

$$\begin{aligned} \text{Then } NaNb &= N(aN)b \\ &= N(Nab) \text{ since } aN = Na \\ &= NNab \\ &= Nab \text{ (since } NN = N) \end{aligned}$$

Conversely suppose that the product of any two right cosets of N is again a right coset of N . Then $NaNb$ is a right coset of N . Further $ab = (ea)(eb)$

$\in NaNb$.

Hence $NaNb$ is the right coset containing ab .

$\therefore NaNb = Nab$

Now we prove that N is a normal subgroup of G .

Let $a \in G$ and $n \in N$.

Then $ana^{-1} = eana^{-1} \in NaNa^{-1} = Naa^{-1} = N$

$\therefore ana^{-1} \in N$.

Hence N is a normal subgroup of G .

Thm : Let N be a normal subgroup of a group G . Then G/N is a group under the operation defined by $NaNb = Nab$.

Pf : By the previous thm, the operation given by $NaNb = Nab$ is a well defined binary operation in G/N .

Now let $Na, Nb, Nc \in G/N$.

Then $Na (NbNc) = Na (Nbc)$

$$= Na (bc)$$

$$= N(ab) c$$

$$= (NaNb) Nc.$$

The binary operation is associative.

$$Ne = Ne \in G/N \text{ and } NaNe = Nae = Na = NeNa$$

$\therefore Ne$ is the identity element.

$$\text{Also } NaNa^{-1} = Naa^{-1} = Ne = Na^{-1}Na$$

$\therefore Na^{-1}$ is the inverse of Na .

$\therefore G/N$ is a group.

Definition :

Let N be a normal subgroup of G . Then the group G/N is called the quotient group (factor group) of G modulo N .

Examples :

$3\mathbb{Z}$ is a normal subgroup of $(\mathbb{Z}, +)$. The quotient group $\mathbb{Z}/3\mathbb{Z} = [3\mathbb{Z} + 0, 3\mathbb{Z} + 1, 3\mathbb{Z} + 2]$ Hence $\mathbb{Z}/3\mathbb{Z}$ is a group of order 3.

UNIT - 4

4.1 Homomorphism of groups

Definition : Homomorphism into :

A mapping f from a group G into a group G' is said to be a homomorphism of G into G' if

$$f(a,b) = f(a) f(b) \quad \forall a,b \in G.$$

Homomorphism onto :

A mapping f from a group G onto a group G' is said to be a homomorphism of G onto G' if

$$f(ab) = f(a) f(b) \quad \forall a,b \in G.$$

Also then G' is said to be a homomorphic image of G .

Endomorphism :

A homomorphism of a group into itself is called an endomorphism.

Example -1: Show that the mapping f of the symmetric group P_n onto the multiplicative group $G' = \{1, -1\}$ is defined by

$$f(x) = 1 \quad (\text{or}) \quad -1$$

according as α is an even (or) odd permutation in P_n is a homomorphism of P_n onto G' .

Solution : We know that the product of two permutations both even (or) both odd is even while the product of one even (or) one odd permutation is odd we shall show that

$$f(\alpha, \beta) = f(\alpha) f(\beta) \quad \forall \alpha, \beta \in P_n$$

(i) If α, β are both even, then

$$f(\alpha, \beta) = 1 = 1 \cdot 1 = f(\alpha) f(\beta)$$

(ii) If α, β are both odd, then

$$f(\alpha, \beta) = 1 = (-1) (-1) = f(\alpha) f(\beta)$$

(iii) If α is odd and β is even, then

$$f(\alpha, \beta) = -1 = (-1) (1) = f(\alpha) f(\beta)$$

(iv) If α is even and β is odd then

$$f(\alpha, \beta) = -1 = (1) (-1) = f(\alpha) f(\beta)$$

$$f(\alpha, \beta) = -1 = (1)(-1) = f(\alpha) f(\beta)$$

$$\text{Thus } f(\alpha, \beta) = f(\alpha) f(\beta) \quad \forall \alpha, \beta \in P_n$$

Also obviously f is onto G' . Therefore f is a homomorphism of P_n onto G' .

Example : Let G be a group and let ' e ' be the identity element of G . Then the mapping $f : G \rightarrow G$ defined by $f(a) = e \quad \forall a \in G$ is an endomorphism of G .

Solution : Let a, b be any two elements of G .

$$\text{Then } f(a) = e; \quad f(b) = e$$

$$\text{Now } f(ab) = e = ee = f(a) f(b).$$

Thus f is a homomorphism of G into G . Therefore f is an endomorphism of G .

Theorem : If f is a homomorphism of a group G into a group G' then

(i) $f(e) = e^1$ where e is the identity of G and e^1 is identity of G' .

(ii) $f(a^{-1}) = [f(a)]^{-1} \quad \forall a \in G$.

(iii) If the order of $a \in G$ is finite then the order of $f(a)$ is a divisor of the order of a .

Pf : (i) Let $a \in G$ then $f(a) \in G'$ we have

$$f(a) e^1 = f(a)$$

$$= f(ae)$$

$$= f(a) f(e)$$

Now G' is a group. Therefore

$$f(a) e^1 = f(a) f(e).$$

$$\Rightarrow e^1 = f(e) \quad (\text{by left cancellation Law})$$

(ii) Let a be any element of G . Then $a^{-1} \in G$.

$$\text{We have } e^1 = f(e) = f(aa^{-1}) = f(a) f(a^{-1})$$

$$\therefore f(a^{-1}) \text{ is the inverse of } f(a) \text{ in the group } G'. \quad \text{Thus } f(a^{-1}) = [f(a)]^{-1}$$

(iii) Let $a \in G$ and $O(a) = m$ we have $O(a) = m \Rightarrow a^m = e$

$$\therefore f(a^m) = f(e)$$

$$\Rightarrow f(a \dots m \text{ times}) = e^1.$$

$$\Rightarrow [f(a)]^m = e^1.$$

\therefore If n is the order of $f(a)$ in G' , then ' n ' must be a divisor of m .

Kernal of a Homomorphism :

Definition : If f is a homomorphism of a group G into a group G' , then the set K of all those elements of G which are mapped by f onto the identity e' of G' is called the kernal of homomorphism.

Thus if f is a homomorphism of G into G' then K is the kernel of f if $K = \{x \in G / f(x) = e' \text{ where } e' \text{ is identity of } G'\}$

Theorem - 1 :

If f is a homomorphism of a group G into a group G' with kernel K then K is a normal subgroup of G .

Pf : Let f be a homomorphism of a group G into a group G' . Let e, e' be the identities of G and G' respectively.

Let K be the Kernel of f .

Then $K = \{x \in G : f(x) = e'\}$.

Since $f(e) = e'$ therefore atleast $e \in K$.

Thus K is not empty.

Let $a, b \in K$ then $f(a) = e' : f(b) = e'$

we have $f(ab^{-1}) = f(a) f(b^{-1}) = f(a) f(b)^{-1} = e' e'^{-1} = e'$.

$\therefore ab^{-1} \in K$.

Thus $a : b \in K \Rightarrow ab^{-1} \in K$

Therefore K is a subgroup of G . Now to prove that K is normal in G . Let g be any element of G and k be any element of K .

Then $f(k) = e'$ we have

$$f(gk^{-1}g) = f(g) f(k) f(g^{-1})$$

$$= f(g) e' [f(g)]^{-1}$$

$$= f(g) [f(g)]^{-1} = e'$$

$$gkg^{-1} \in K.$$

Thus $g \in G : k \in K \Rightarrow gkg^{-1} \in K$.

$\therefore K$ is a normal homomorphism of G .

Thm : Let f be a homomorphism of a group G into a group G' with kernel K . Let ' a ' be a given element of G such that $f(a) = a' \in G'$. Then the set of all those elements of G which have the image a' in G' is the coset Ka of K in G .

Pf : Let e, e' be the identities of G and G' respectively. Let $a \in G$ and $f(a) = a' \in G'$

$$\text{Let } f^{-1}(a') = \{x \in G / f(x) = a'\}$$

Then to prove that $f^{-1}(a') = Ka$

Let $y \in Ka$ Then $y = ka$ for some $k \in K$.

We have $f(y) = f(ka) = f(k) f(a)$

$$= e' f(a)$$

$$= f(a) = a'$$

$$\therefore y \in f^{-1}(a')$$

Thus $y \in Ka \Rightarrow y \in f^{-1}(a')$

$$\therefore Ka \subseteq f^{-1}(a') \dots\dots\dots(1)$$

Now let z be any element of $f^{-1}(a')$ then $f(z) = a'$

$$\begin{aligned} \text{We have } f(za^{-1}) &= f(z) f(a^{-1}) = f(z) [f(a)]^{-1} \\ &= a' (a')^{-1} = e' \end{aligned}$$

$$\therefore za^{-1} \in K.$$

$$\Rightarrow (za^{-1})a \subseteq Ka$$

$$\Rightarrow z \in Ka$$

Thus $z \in f^{-1}(a') \Rightarrow z \in Ka$

$$\therefore f^{-1}(a') \subseteq Ka \dots\dots\dots(2)$$

From (1) and (2) we get

$$f^{-1}(a') = Ka.$$

Thm : The necessary and sufficient condition for a homomorphism ' f ' of a group G into a group G' with kernel K to be an isomorphism of G into G' is that $K = \{e\}$.

Pf : Let f be a homomorphism of a group G into a group G' . Let e, e' be the identities of G, G' respectively. Let K be the kernel of f .

Suppose f is an isomorphism of G into G' . Then f is one-one.

Let $a \in K$ then $f(a) = e^1$ (by def of kernel)

$$\Rightarrow f(a) = f(e) \quad (\because f(e) = e^1)$$

$$\Rightarrow a = e \quad (\because f \text{ is one-one})$$

Thus $a \in K \Rightarrow a = e$. In otherwords 'e' is the only element of G which belongs to K . Therefore $K = \{e\}$.

Conversely suppose that $K = \{e\}$. Then to prove that f is an isomorphism of G into G^1 . (ie) to prove that f is 1-1.

If $a, b \in G$ then $f(a) = f(b)$

$$\Rightarrow f(a) [f(b)]^{-1} = f(b) [f(b)]^{-1}$$

$$\Rightarrow f(a) f(b^{-1}) = e^1 \quad (\because f \text{ is a homomorphism}).$$

$$\Rightarrow f(ab^{-1}) = e^1$$

$$\Rightarrow ab^{-1} \in K.$$

$$\Rightarrow ab^{-1} = e$$

$$\Rightarrow ab^{-1}b = eb$$

$$\Rightarrow a = b$$

$\therefore f$ is one-one.

Hence f is an isomorphism of G into G' .

Thm : Suppose G is a group and N is a normal subgroup of G . Let f be a map mapping from G to G/N defined by

$$f(x) = Nx \quad \forall x \in G.$$

Then f is a homomorphism of G onto G/N and kernel $f = N$.

Pf : Consider the mapping

$f: G \rightarrow G/N$ such that $f(x) = Nx \quad \forall x \in G$. Let Nx be any element of G/N . Then $x \in G$ we have $f(x) = Nx$. Therefore the mapping f is onto G/N .

Let $a, b \in G$. Then

$$f(a, b) = Nab = (Na) (Nb)$$

$$= f(a) f(b)$$

$\therefore f$ is a homomorphism of G onto G/N .

Thus every quotient group of a group is a homomorphic image of the group. The mapping $f:G \rightarrow G/N$ such that $f(x) = Nx \forall x \in G$ is called as natural mapping of G onto G/N .

Let K be the kernel of this homomorphism f . The identity of the quotient group G/N is the coset N . So $K = \{y \in G / f(y) = N\}$.

We shall prove that $K = N$.

Let $k \in K$ Then $f(k) = N$ (ie) identity of G/N .

But by def of f , we have $f(k) = Nk$

Now $Nk = N \Rightarrow K \in N$.

Thus $k \in K \Rightarrow k \in N$ Therefore $K \subseteq N$.

Again Let n be any element of N .

Then $Nn = N$.

We have $f(n) = Nn = N$.

Therefore $n \in K$.

Thus $n \in N \Rightarrow n \in K$.

$\therefore N \subseteq K$.

Consequently $K = N$.

4.2 Fundamental theorem on homomorphism of groups

Every homomorphic image of a group G is isomorphic to some quotient group of G .

Pf : Let G' be the homomorphic image of a group G and f be the corresponding homomorphism. Then f is a homomorphism of G onto G' . Let K be the kernel of this homomorphism. Then K is a normal subgroup of G . We shall prove that $G/K \cong G'$.

If $a \in G$, then $ka \in G/K$ and $f(a) \in G'$, consider the mapping $\phi: G/K \rightarrow G'$ such that $\phi(ka) = f(a) \forall a \in G$.

First we shall show that the mapping ϕ is well defined. (ie) If $a, b \in G$ and $ka = kb$ then $\phi(ka) = \phi(kb)$.

We have $ka = kb$

$$\Rightarrow ab^{-1} \in K.$$

$$\Rightarrow f(ab^{-1}) = e' \text{ (identity of } G')$$

$$\Rightarrow f(a) f(b^{-1}) = e^1 \Rightarrow f(a) [f(b)]^{-1} = e^1$$

$$\Rightarrow f(a) [f(b)^{-1}] f(b) = e^1 f(b)$$

$$\Rightarrow f(a) e^1 = f(b)$$

$$\Rightarrow f(a) = f(b)$$

$$\Rightarrow \phi(ka) = \phi(kb)$$

$\therefore \phi$ is well defined.

ϕ is one-one. We have

$$\phi(ka) = \phi(kb)$$

$$\Rightarrow f(a) = f(b)$$

$$\Rightarrow f(a) [f(b)]^{-1} = f(b) [f(b)]^{-1}$$

$$\Rightarrow f(a) f(b)^{-1} = e^1$$

$$\Rightarrow f(ab^{-1}) = e^1$$

$$\Rightarrow ab^{-1} \in K \quad (\because k \text{ is kernel})$$

$$\Rightarrow ka = kb.$$

$\therefore \phi$ is one-one.

ϕ is onto G' :

Let y be any element of G' . Then $y = f(a)$ for some $a \in G$ because f is onto G' .

Now $ka \in G/K$ and we have $\phi(ka) = f(a) = y$.

$\therefore \phi$ is onto G' .

Finally we have $\phi(ka) \phi(kb) = \phi(kab)$

$$= f(ab)$$

$$= f(a) f(b)$$

$$= \phi(ka) \phi(kb)$$

$\therefore \phi$ is an isomorphism of G/K onto G' .

Hence $G/K \cong G'$

Examples :

1) Let f be a homomorphic mapping of a group G into group G' . Let $f(G)$ be the homomorphic image of G in G' . Then $f(G)$ is a subgroup of G' .

Pf : We have $f(G) = \{f(x) / x \in G\}$. Obviously $f(G) \subseteq G'$. Let a^1, b^1 be any two elements of $f(G)$. Then $f(a) = a^1$ $f(b) = b^1$ for some $a, b \in G$. We have

$$\begin{aligned} a^1(b^1)^{-1} &= f(a) (f(b))^{-1} = f(a) f(b^{-1}) \\ &= f(ab^{-1}) \in f(G) \end{aligned}$$

Since $ab^{-1} \in G$.

Thus $a^1, b^1 \in f(G) \Rightarrow a^1 (b^1)^{-1} \in f(G)$

$\therefore f(G)$ is a subgroup of G'

2) Show that every homomorphic image of an abelian group is abelian and converse is not true.

Solution : Let G be an abelian group. Let f be a homomorphic mapping of G onto a group G' . Then G' is a homomorphic image of G .

Let a^1, b^1 be any two elements of G' . Then $f(a) = a^1$ $f(b) = b^1$ for some $a, b \in G$. We have

$$\begin{aligned} a^1, b^1 &= f(a) f(b) = f(ab) \\ &= f(ba) \\ &= f(b) f(a) = b^1 a^1 \end{aligned}$$

$\therefore G'$ is abelian.

The converse is not true P_3 is a non-abelian group. A_3 is a normal subgroup of P_3 . The quotient group P_3/A_3 is a homomorphic image of P_3 . Now P_3/A_3 is of order 2 and is abelian.

3) Show that a homomorphism from a simple group is either (or) one-to-one.

Solution : Let G be a simple group and f be a homomorphism of G into another group G' . Then $\ker f$ is a normal subgroup of G . But the only normal subgroups of the simple group G are G itself and $\{e\}$. Therefore either $\ker f = G$ (or) $\ker f = \{e\}$. If $\ker f = G$, the f -image of each element of G is the identity of G' and so the homomorphism f is a trivial one. If $\ker f = \{e\}$ the homomorphism f is one-to-one. Hence the result.

4) If f is a homomorphism of G onto G' and g a homomorphism of G' onto G'' . Show that $g \circ f$ is a homomorphism of G onto G'' . Also show that the kernel of f is a subgroup of that of $g \circ f$.

Pf : f is a mapping of G onto G' and g is a mapping of G' onto G'' . Therefore $g \circ f$ is a mapping of G onto G'' and we have

$$(g \circ f)(x) = g(f(x)) \quad \forall x \in G.$$

Let $a, b \in G$ Then

$$\begin{aligned} (g \circ f)(ab) &= g[f(ab)] \\ &= g[f(a)f(b)] \quad (\because f \text{ is a homomorphism}) \\ &= g[f(a)]g[f(b)] \quad (\because g \text{ is a homomorphism}) \\ &= [(g \circ f)(a)][(g \circ f)(b)] \end{aligned}$$

$\therefore g \circ f$ is a homomorphism of G onto G'' . Let K be the kernel of $g \circ f$. Then $K = \{y \in G \mid (g \circ f)(y) = e'' \text{ where } e'' \text{ is the identity of } G''\}$.

Let K' be the kernel of f . Then $K' = \{z \in G \mid f(z) = e' \text{ where } e' \text{ is the identity of } G'\}$.

Both K and K' are normal subgroups of G . In order to show that K' is a subgroup of K it is sufficient to prove that $K' \subseteq K$.

$$\text{Let } K' \subseteq K' \text{ then } f(k') = e'$$

$$\text{Also } (g \circ f)(k') = g(f(k')) = g(e') = e''$$

$$\therefore k' \in K$$

$$\text{Thus } k' \in K' \Rightarrow k' \in K$$

$$\Rightarrow K' \subseteq K$$

5) Let G be the multiplicative group of all $n \times n$ non singular matrices with elements as real numbers and let G' be the multiplicative group of all non-zero real numbers. Show that the mapping $f: G \rightarrow G'$ such that $f(A) = |A| \quad \forall A \in G$ is a homomorphism of G onto G' what is the kernel of this homomorphism?

Pf : Let A, B be any two $n \times n$ non-singular matrices with elements as real numbers. We have

$$f(AB) = |AB| = |A| |B| = f(A) f(B).$$

Also if x is any non-zero real number then there exists an $n \times n$ matrix $A \in G$ whose determinant is equal to x .

f is a homomorphism of G onto G' . The identity of G' is 1. Therefore the kernel of f is a subgroup of G consisting of matrices with determinant equal to 1.

4.3 Cayley's Theorem

Every finite group is isomorphic to a permutation group.

Pf : Let $G = \{a_1, a_2, \dots, a_n\}$ be a finite group of order n .

If $a \in G$, then the n products aa_1, aa_2, \dots, aa_n are all distinct for if possible.

Let $aa_j = aa_k \dots \dots \dots (1)$

Then $a_j = a_k$ by cancellation law. This is a contradiction since $a_j \neq a_k$. Hence $aa_j \neq aa_k$.

Therefore the n products given in (1) are the n elements of G in some order. It follows that the mapping

$f_a : G \rightarrow G : f_a(x) = ax : a \in G$ is one-one and onto.

Hence $f_a = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ aa_1 & aa_2 & \dots & aa_n \end{pmatrix}$ is a permutation on n symbols.

It is evident that there will be n permutations like f_a . Let the set of these n permutations like f_a . Let the set of these n permutations be denoted by G' . Then $G' = \{f_a / a \in G\}$.

First we shall show that G' is a group with respect to the operation known as composite (or) product of two functions.

Closure Property : Let $f_a, f_b \in G'$ where $a, b \in G$ from our definition of product of two functions we have

$$\begin{aligned} (f_a f_b)(x) &= f_a(f_b(x)) = f_a(bx) \\ &= a(bx) = (ab)x \\ &= f_{ab}(x) \text{ for all } x \in G. \end{aligned}$$

Therefore by the definition of equality of two functions. We have

$$f_a f_b = f_{ab} \dots \dots \dots (1)$$

Since $ab \in G$. $f_{ab} \in G'$ and thus G' is closed with respect to the product of functions.

Associativity : Let $f_a, f_b, f_c \in G'$ where $a, b, c \in G$. Then

$$f_a(f_b f_c) = f_a f_{bc} \quad [\because \text{from (1) } f_b f_c \text{ from (1)}]$$

$$\begin{aligned}
&= f_{a(bc)} \\
&= f_{(ab)} c \quad (\text{by associativity in } G) \\
&= f_{ab} f_c \quad (\text{from (1)}) \\
&= (f_a f_b) f_c \quad (\text{from (1)})
\end{aligned}$$

Therefore the operation in G' is associative.

Existence of Identity :

If e is the identity of G then f_e is the identity of G' because of every f_a in G' .
We have

$$\begin{aligned}
f_e f_a &= f_{ea} = f_a \\
f_a f_e &= f_{ae} = f_a
\end{aligned}$$

Existence of Inverse :

If a^{-1} is the inverse of a in G then $f_{a^{-1}}$ is the inverse of f_a in G' because

$$\begin{aligned}
f_{a^{-1}} f_a &= f_{a^{-1}a} = f_e \\
f_a f_{a^{-1}} &= f_{aa^{-1}} = f_e
\end{aligned}$$

Thus G' is a group.

Now we shall show that $G \cong G'$. Consider the function ϕ from G into G' defined by $\phi(a) = f_a \quad \forall a \in G$.

ϕ is one-one :

If $a, b \in G$. Then

$$\begin{aligned}
\phi(a) = \phi(b) &\Rightarrow f_a f_b \\
&\Rightarrow f_a(x) f_b(x) \quad \forall x \in G \\
&\Rightarrow ax = bx \quad \forall x \in G \\
&\Rightarrow a = b.
\end{aligned}$$

$\therefore \phi$ is one-one.

ϕ is onto :

Let f_a be any element of G' . Then $a \in G$ and we have $\phi(a) = f_a$. Therefore ϕ is onto.

ϕ preserves compositions in G and G' :

If $a, b \in G$ then

$$\phi(a, b) = f_{ab} \quad (\text{by def of } \phi)$$

$$= f_a f_b \quad (\text{from (1)})$$

$$= \phi(a) \phi(b) \quad (\text{by def of } \phi)$$

$\therefore \phi$ preserves compositions in G and G' .

$$\therefore G \cong G'.$$

UNIT - 5

RING

5.1. Ring-definition and examples

Suppose R is a non-empty set equipped with two binary operations called addition and multiplication and denoted by '+' and '.' respectively (ie) for all $a, b \in R$. We have $a + b \in R$ and $a \cdot b \in R$. Then this algebraic structure $(R, +, \cdot)$ is called a ring if the following postulates are satisfied.

1. Addition is associative.

$$(ie) (a + b) + c = a + (b + c) \quad \forall a, b, c \in R.$$

2. Addition is commutative (ie)

$$a + b = b + a \quad \forall a, b \in R.$$

3. There exists an element denoted by 0 in R such that $0 + a = a \quad \forall a \in R$.

4. To each element a in R there exists an element $-a$ in R such that $(-a) + a = 0$

5) Multiplication is associative.

$$To \ a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in R$$

6) Multiplication is distributive with respect to addition (ie) for all a, b, c in R .

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{Left distributive law.}$$

$$(b + c) \cdot a = b \cdot a + c \cdot a \quad \text{Right distributive law.}$$

Since addition is commutative in R therefore we shall have $0 \in R$ such that $0 + a = a = a + 0 \quad \forall a \in R$.

Also if $a \in R$. then we shall have $(-a) + a = 0 = a + (-a)$

Thus R will be an abelian group with respect to addition the element $0 \in R$ will be the additive identity. It is called the zero element of the Ring. Since in a group the identity element is unique, therefore everything will possess unique zero element and it will be the identity element for addition composition. We shall always denote this element by the symbol zero. It is the symbol which will represent the additive identity of the ring.

In a ring every element will possess a unique inverse for addition composition we shall denote the additive inverse of ' a ' by the symbol ' $-a$ ' we shall define $a - b = a + (-b)$

The equation $a + x = b$ will have a unique solution in R and it will be $x = b - a$ obviously $a + (b - a) = a + [b + (-a)]$

$$= a + [(-a) + b]$$

$$= [a + (-a)] + b$$

$$= 0 + b = b.$$

similarly the equation $y + a = b$ will have a unique solution in R and it will be $y = b - a$.

Both the cancellation laws will hold good for addition in R .

(ie) for all a, b, c in R

$$a + b = a + c$$

$$\Rightarrow b = c \text{ and } b + a = c + a$$

$$\Rightarrow b = c$$

If in a ring we have $a + b = 0$ then $a = -b$ and $b = -a$.

Ring with unity:

If in a ring R there exists an element denoted by 1 such that $1 \cdot a = a = a \cdot 1$ $\forall a \in R$ then R is called a ring with unit element. The element $1 \in R$ is called the unit element of the ring obviously 1 is the multiplicative identity of R thus if a ring possess multiplicative identity then it is a ring with unity:

Commutative ring

If in a ring R , the multiplication composition is also commutative (ie) if we have $ab = ba$ $\forall a, b \in R$ then R is called a commutative ring.

Elementary prospectus of a Ring

Theorem: If R is a ring then for all $a, b, c \in R$.

$$(i) a0 = 0a = 0$$

$$(ii) a(-b) = -(ab) = (-a)b$$

$$(iii) (-a)(-b) = ab$$

$$(iv) a(b-c) = ab - ac$$

$$(v) (b-c)(a) = ba - ca$$

Proof: (i) we have

$$a0 = a(0+0)$$

$$= a0 + a0 \text{ (by left distributive law)}$$

$$\therefore 0 + a0 = a0 + a0 ; \therefore a0 \in R \text{ and } 0 + a0 = a0$$

Now R is a group with respect to addition therefore applying right cancellation law for addition in R we get $0 = a$ similarly we have $0a = (0+0)a = 0a + 0a$

$$\therefore 0 + aa = 0a + oa \text{ (since } 0 + 0a = 0a \text{)}$$

Applying right cancellation law for addition in R. We get $0 = 0a$

$$\text{(iii) we have } a [(-b) + b] = a0$$

$$\Rightarrow a(-b) + ab = 0 \text{ (by using left distributive law and the result (i))}$$

$$\Rightarrow a(-b) = -(ab) \text{ since in a ring } a + b = 0 \Rightarrow a = -b$$

similarly we have $(-a + a) b = 0b$.

$$\Rightarrow (-a) b + ab = 0$$

$$\Rightarrow (-a)b = -(ab) \text{ since in a ring } a + b = 0 \Rightarrow a = -b$$

$$\text{(iii) we have } (-a) (-b) = - [(-a)b]$$

$$\text{since } a(-b) = -(ab)$$

$$= - [-(ab)]$$

$$\text{since } (-a) b = -(ab)$$

$$= ab$$

since R is a group with respect to addition and in a group we have $-(-a) = a$

$$\text{(iv) we have } a(b-c) = a [b + (-c)]$$

$$= ab + a(-c)$$

$$= ab + [-(ac)]$$

$$= ab - ac$$

(v) we have

$$(b-c)a = [b + (-c)] a$$

$$= ba + (-c)a \text{ (right distributive law)}$$

$$= ba + [-(ca)]$$

$$= ba - ca$$

Examples of Rings

1. The set R consisting of a single element zero with two binary operations defined by $0 + 0 = 0$ and $0 \cdot 0 = 0$ is a ring. This ring is called the null ring (or) the zero ring.

2. The set I of all integers is a ring with respect to addition and multiplication of integers as the two ring compositions. This ring is called the ring of integers.

Pf. As in groups we should first prove that I is an abelian group with respect to addition of integers. Further we observe that

(i) The product of two integers is also an integer. Therefore I is closed with respect to multiplication of integers.

(ii) Multiplication of integers is an associative composition.

(iii) Multiplication of integers is distributive with respect to addition of integers

(ie) if a, b, c are any elements of I ,

$$\text{then } a(b+c) = ab + ac$$

$$\text{and } (b+c)a = ba + ca$$

Therefore I is a ring with respect to addition and multiplication of integers. The integer zero is the zero element of this ring. Also the multiplicative identity exists and is the integer one. We have $1a = a = a1 \forall a \in I$. Thus the ring of integer is a ring with unity. The integer 1 is the unit element of this ring.

The multiplication of integers is a commutative composition. Therefore it is also a commutative ring.

Rings with (or) without zero divisors

We have proved that in any ring R , if 0 is the additive identity. (ie) the zero element of the ring, then $0a = a0 = 0 \forall a \in R$. However there are rings in which it is possible that $ab = 0$ when neither $a = 0$ nor $b = 0$ such elements are called zero divisors.

Definition:

A non zero element of a ring R is called a zero divisor (or) a divisor of zero. If there exists an element $b \neq 0 \in R$ such that either $ab = 0$ (or) $ba = 0$

Rings without zero divisors

A ring R is without zero divisors if the product of no two non zero elements of R is zero. (ie) if $ab = 0 \Rightarrow a = 0$ (or) $b = 0$.

On the otherhand if in a ring R there exists non zero elements a and b such that $ab = 0$ then R is said to be a ring with zero divisors.

Cancellation laws in a ring

If R is a ring then R is an abelian group with respect to addition. For addition composition laws hold in all rings. Therefore the question of cancellation laws holding in a ring arises only for the multiplication composition.

We say that cancellation laws hold in a ring R if $a \neq 0$, $ab = ac \Rightarrow b = c$ and $a \neq 0$, $ba = ca \Rightarrow b = c$ where $a, b, c \in R$.

Theorem:

A ring R is without zero divisors \Leftrightarrow and only if the cancellation laws hold in R (ie)

R is without zero divisors \Leftrightarrow Cancellation laws hold in R .

Pf. First suppose that R has no zero divisors. Let a, b, c be any three elements of R such that $a \neq 0$, $ab = ac$ we have $ab = ac \Rightarrow ab - ac = 0$

$$\Rightarrow a(b-c) = 0$$

since R is without zero divisors, therefore $a(b-c) = 0$ and $a \neq 0 \Rightarrow b-c = 0$

$$(ie) \quad b = c$$

Thus the left cancellation law holds in R . Similarly we can show that the right cancellation law holds in R . Conversely suppose that the cancellation laws holds in R . If possible let $ab = 0$ $a \neq 0$, $b \neq 0$.

Then we have $ab = a0$ since $a0 = 0$ Now $a \neq 0$ $ab = a0 \Rightarrow b = 0$ by left cancellation law.

Thus we get a contradiction.

Hence R is without zero divisors.

5.2 Integral domains fields : Division

A ring is called an integral domain if it (i) is commutative (ii) has unit element (iii) is without zero divisors.

The most important example of an integral domain is the ring I of integers. We have proved that " I " is a commutative ring with unity. Also " I " does not possess zero divisors we know that if a, b are integers such that $ab = 0$ then either a (or) b must be zero.

Field : Definition:

A ring R with atleast two elements is called a field if it (i) is commutative (ii) has unity (iii) is such that each non zero element possesses multiplicative inverse.

Example : The rings of real numbers and complex numbers are also examples of fields.

Division ring or skew field:

Definition

A ring R with atleast two elements is called a division ring (or) a skew field if it (i) has unity (ii) is such that each non zero element possesses multiplicative inverse.

Thus a commutative division ring is a field.

Every field is also a division ring. But a division ring is a field if it is also commutative. We shall later on give an example of a skew field which is not commutative (ie) which is not a field.

Theorem : Every field is an integral domain.

Pf: Since a field F is a commutative ring with unity, therefore in order to show that every field is an integral domain. We should show that a field has no zero divisors. Let a, b be elements of F with $a \neq 0$ such that $ab = 0$.

Since $a \neq 0$, a^{-1} exists and we have

$$ab = 0 \Rightarrow a^{-1}(ab) = a^{-1}(0)$$

$$\Rightarrow (a^{-1}a)b = 0$$

$$\Rightarrow b = 0 \quad (\because a^{-1}a = 1)$$

$$\Rightarrow b = 0$$

Similarly, Let $ab = 0$ and $b \neq 0$.

Since $b \neq 0$, b^{-1} exists and we have

$$ab = 0 \Rightarrow (ab)b^{-1} = 0b^{-1}$$

$$\Rightarrow a(bb^{-1}) = 0$$

$$\Rightarrow a = 0$$

$$\Rightarrow a = 0$$

Thus, if $ab = 0$ then $a = 0$ or $b = 0$.

$$\Rightarrow a = 0 \text{ or } b = 0$$

Therefore a field has no zero divisors. Therefore every field is an integral domain. But the converse is not true (ie) every integral domain is not a field. For example, the ring of integers is an integral domain and it is not a field. The only invertible elements of the ring of integers are 1 and -1.

Remark:

A field has no zero divisors, therefore in a field the product of two non-zero elements will again be a non-zero element. Also the unit element $1 \neq 0$ and each non-zero element possesses a multiplicative inverse which is again a non-zero element. The multiplication is commutative as well as associative. Therefore the non-zero elements of a field form an abelian group with respect to multiplication.

Theorem : A skew field (Division ring) has no divisors of zero.

Pf: Let D be a skew field. Then D is a ring with unit element one and each non zero element of D possesses multiplicative inverse. Let a, b be elements of D with $a \neq 0$ such that $ab = 0$.

Since $a \neq 0$ a^{-1} exists and we have

$$ab = 0 \Rightarrow a^{-1}(ab) = a^{-1}(0)$$

$$\Rightarrow (a^{-1}a)(b) = 0$$

$$\Rightarrow 1(b) = 0 \Rightarrow b = 0$$

Similarly let $ab = 0$ with $b \neq 0$.

Since $b \neq 0$, b^{-1} exists and we have

$$ab = 0 \Rightarrow (ab)b^{-1} = 0(b^{-1})$$

$$\Rightarrow a(bb^{-1}) = 0$$

$$\Rightarrow a1 = 0$$

$$\Rightarrow a = 0$$

Therefore a skewfield has no zero divisors.

Theorem :

A finite commutative ring without zero divisors is a field. (or) Every finite integral domain is a field.

Pf.

Let D be a finite commutative ring without zero divisors having n elements a_1, a_2, \dots, a_n . In order to prove that D is a field. We must produce an element $1 \in D$ such that $1a = a \forall a \in D$. Also we should show that for every element $a \neq 0 \in D$ there exists an element $b \in D$ such that $ba = 1$

Let $a \neq 0 \in D$. consider the n products aa_1, aa_2, \dots, aa_n

All these are elements of D . Also they are all distinct for suppose

$$aa_i = aa_j \text{ for } i \neq j$$

$$\text{then } aa_i - aa_j = 0$$

$$\Rightarrow a(a_i - a_j) = 0 \text{ ----- (1)}$$

Since D is without zero divisors and $a \neq 0$, therefore (1) $\Rightarrow a_i - a_j = 0$

$$\Rightarrow a_i = a_j$$

which is $\Rightarrow \Leftarrow$ to $i \neq j$

$\therefore aa_1, aa_2, \dots, aa_n$ are all the n distinct elements of D placed in some order. So one of these elements will be equal to a . Thus there exists an element say $c \in D$ such that $ac = a = ca$ ($\because D$ is commutative) we shall show that this element c is the multiplicative identity of D .

Let y be any element of D . Then from the above discussion for some $x \in D$ we shall have $ax = y = xa$.

$$\text{Now } cy = c(ax)$$

$$= ax \quad (\because ca = a)$$

$$= y \quad (\because ax = y)$$

$$= yc \quad (\because D \text{ is commutative})$$

Thus $cy = y = yc \quad \forall y \in D$. Therefore c is the unit element of the ring D and let us denote it by 1 .

Now $i \in D$ therefore from the above discussion one of the n products aa_1, aa_2, \dots, aa_n will be equal to 1 . Thus there exists an element say $b \in D$ such that

$$ab = 1 = ba$$

$\therefore b$ is the multiplicative inverse of the non-zero element $a \in D$. Thus every non-zero element of D is invertible.

Hence D is a field.

Definition for invertible

If R is a ring with unity, then an element $a \in R$ is called invertible if there exists $b \in R$ such that $ab = 1 = ba$. Also we can write $b = a^{-1}$.

Examples:

1) If a, b, c, d are elements of a ring R , then evaluate $(a+b)(c+d)$

Pf. We have

$$(a+b)(c+d) = a(c+d) + b(c+d) \quad (\text{by right distributive law})$$

$$= ac + ad + bc + bd \quad (\text{by left distributive law}).$$

2) Prove that if $a, b \in R$ then $(a+b)^2 = a^2 + ab + ba + b^2$ where by x^2 by mean xx .

Pf: we have

$$(a+b)^2 = (a+b)(a+b)$$

$$= a(a+b) + b(a+b) \quad (\text{by right distributive law})$$

$$= (aa + ab) + (ba + bb) \quad (\text{by left distributive law})$$

$$= a^2 + ab + ba + b^2$$

3) If a, b, c, d are any elements of a ring R . Prove that

$$(a-b)(c-d) = (ac+bd) - (ad+bc)$$

Pf we have

$$(a-b)(c-d) = (a-b)c - (a-b)d$$

$$(\because a(b-c) = ab-ac)$$

$$= (ac-bc) - (ad-bd)$$

$$= (ac-bc) - ad + bd$$

$$= (ac+ba) - bc - ad.$$

since addition is commutative and associative.

$$= (ac+bd) - (bc+ad).$$

4) If R is a system satisfying all the conditions for a ring with unit element with the possible exception of $a+b = b+a$. Prove that the axiom $a+b = b+a$ must hold in R and that R is thus a ring.

Pf: Since 1 is an element of R .

We have $(a+b)(1+1) = a(1+1) + b(1+1)$ (by right distributive law)

$$= (a1+a1) + (b1+b1)$$

$$= (a+a) + (b+b) \text{ ----- (1)}$$

Also $(a+b)(1+1) = (a+b)1 + (a+b)1$ (by left distributive)

$$= (a+b) + (a+b) \text{ ----- (ii)}$$

(since 1 is the unit element)

From (i) and (ii) we get

$$(a+a) + (b+b) = (a+b) + (a+b).$$

$$\Rightarrow [(a+a) + b] + b = [(a+b) + a] + b \text{ (by associativity of addition)}$$

$$\Rightarrow (a+a) + b (a+b) + a \text{ [(by right cancellation law for addition in R.)}$$

Since with the given postulates R is a group with respect to addition)]

$$\Rightarrow a + (a+b) = a + (b+a) \text{ [by associativity of addition in R]}$$

$$\Rightarrow a+b = b+a \text{ (by left cancellation law for addition in R)}$$

Thus addition is commutative in R .

Hence R is a ring.

5) Prove that the set M of 2×2 matrices over the field of real numbers is a ring with respect to matrix addition and multiplication. Is it a commutative ring with unity element? Find the zero element. Does this ring possess zero divisors?

Solution:

Let $A, B \in M$ then $A + B \in M$ and $AB \in M$. Therefore M is closed with respect to addition and multiplication of matrices.

Both addition and multiplication of matrices are associative compositions.

$$\therefore A + (B + C) = (A + B) + C \quad \forall A, B, C \in M.$$

$$\text{and } A(BC) = (AB)C \quad \forall A, B, C \in M.$$

Addition of matrices is a commutative composition. Therefore for all $A, B \in M$ we have $A + B = B + A$.

If O is the null matrix of the type 2×2 . Then $O \in M$ and $O + A = A \quad \forall A \in M$.

Further multiplication of matrices is distributive with respect to addition.

$$(A(B + C)) = AB + AC$$

$$\text{and } (B + C)A = BA + CA \quad \forall A, B, C \in M.$$

$\therefore M$ is a ring with respect to the given compositions.

Multiplication of matrices is not in general a commutative composition.

for example

$$\text{If } A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 3 & 11 \end{pmatrix}$$

$$BA = \begin{pmatrix} 8 & 14 \\ 3 & 5 \end{pmatrix}$$

Thus $AB \neq BA$ and so the ring is a non-commutative ring.

$$\text{If } I \text{ be the unit matrix of the type } 2 \times 2 \text{ (ie) if } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then $I \in M$. Also we have $AI = A = IA \quad \forall A \in M$.

$\therefore I$ is the multiplicative identity.

Thus the ring possesses the unit element and we have $I = 1$ (the unit element of the ring).

The null matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the additive identity and is therefore the zero

element of the ring (ie) $O=0$ (the zero element of the ring). The ring possess zero divisors. For example if

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$$

$$\text{Then } AB = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus the product of two non zero elements of the ring is equal to the zero element of the ring.

Book for Reference:

1. Modern Algebra - Dr. S. Arumugam & others

2. Differential equation by

- Mr. S. Narayanan

- Mr. T.K. Manika Vasagam Pillai.

DIFFERENTIAL EQUATIONS

UNIT - 6

6.1 Equation of the first order

$$\text{Let } \frac{dy}{dx} = p.$$

Introduction

The problem of solving differential equations is a natural goal of differential and integral calculus. Further many of the general laws of nature in Physics, Chemistry, Biology and Astronomy can be expressed in the language of differential equations and hence the theory of differential equations is the most important part of mathematics for understanding physical sciences. Also this theory has many applications in geometry, Economics, Mechanics etc.

Differential equations :-

A differential equation is an equation which involves derivatives.

The following are some examples of differential equations.

$$1. y^1 = \sin x$$

$$2. y^{11} = 3y^1 + 2y = ex$$

$$3. (y^{11})^2 + (y^1)^3 + 3y = x^2$$

$$4. y^{111} + 2(y^{11})^2 + y^1 = ex$$

$$5. y = xy^{11} + r \sqrt{1 + (y^1)^2}$$

$$6. \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = z$$

$$7. \frac{\partial^2 z}{\partial x^2} - x \frac{\partial^2 z}{\partial y^2} = x^2 + y.$$

In a differential equation if there is a single independent variable and the derivatives are ordinary derivatives then it is called an ordinary differential equation.

Examples :-

1 to 5 are ordinary differential equations. If there are two (or) more independent variables and the derivatives are partial derivatives then it is called a partial differential equation.

Examples :-

6 and 7 are partial differential equations. The order of a differential equation is the order of the highest derivative appearing in it.

Examples :-

1 and 6 are of first order and 2, 3, 5, 7 are of order two and 4 is of order 3. The degree of the differential equation is the degree of the highest ordered derivative occurring in it when the differential coefficients are free from radicals and fractions.

In the above examples all except 3 and 5 are of degree one : Examples 3 and 5 are of degree two.

For the present we confine ourselves exclusively to the study of ordinary differential equations. The theory of partial differential equations will be taken in next chapter.

The general ordinary differential equation of n^{th} order is of the form $f(x, y^1, y^{11}, \dots, y^{(n)}) = 0$.

Formation of differential equations :-

Given an equation in the variables x and y containing n arbitrary constants. We differentiate it n times to get n additional equations involving the n arbitrary constants. We eliminate the n arbitrary constants from the above $n + 1$ equations to obtain a differential equation of n^{th} order.

Problem - 1

Form the differential equation for which $xy = ae^x + be^{-x} + x^2$ is a solution.

Solution :-

$$\text{Let } xy = ae^x + be^{-x} + x^2 \quad \text{----- (1)}$$

There are two arbitrary constants in (1). Hence differentiating (1) twice w.r.to x . we get

$$xy^1 + y = ae^x - be^{-x} + 2x \quad \text{----- (2)}$$

$$xy^{11} + 2y^1 = ae^x + be^{-x} + 2 \quad \text{----- (3)}$$

From (1) and (3) we get

$$xy^{11} + 2y^1 = (xy - x^2) + 2 \text{ which is the required differential equation.}$$

6.2. Equation solvable for p, y or x

Type - A :

Equation solvable for $\frac{dy}{dx}$

We shall denote $\frac{dy}{dx}$ hereafter by P.

Let the equation of the first order and of the nth degree be

$$P^n + P_1 P^{n-1} + P_2 P^{n-2} + \dots + P_n = 0 \quad (1)$$

Where P_1, P_2, \dots, P_n denote functions of x and y. Suppose the first member of (1) can be resolved into factors of the first degree of the form.

$$(P - R_1) (P - R_2) \dots (P - R_n)$$

Any relation between x and y which makes any of these factors vanish is a solution of (1).

Let the primitives of

$$P - R_1 = 0, P - R_2 = 0 \quad \text{etc be}$$

$\phi_1(x, y, c_1) = 0 ; \phi_2(x, y, c_2) = 0 \dots \phi_n(x, y, c_n) = 0$ respectively. Where c_1, c_2, \dots, c_n are arbitrary constants. Without any loss of generality we can replace c_1, c_2, \dots, c_n by c where c is an arbitrary constant. Hence the solution of (1) is $\phi_1(x, y, c) \cdot \phi_2(x, y, c) \dots \phi_n(x, y, c) = 0$.

Examples :-

$$\text{Solve } x^2 p^2 + 3xyp + 2y^2 = 0$$

$$\text{Solving for P, } P = -\frac{y}{x} \quad (\text{or}) \quad -\frac{2y}{x}$$

$$\frac{dy}{dx} = -\frac{y}{x} \quad \therefore P = \frac{dy}{dx}$$

$$\Rightarrow xy = c.$$

$$\frac{dy}{dx} = -\frac{2y}{x} \quad \text{gives } yx^2 = c.$$

$$\text{The solution is } (xy - c)(yx^2 - c) = 0.$$

Ex - 2:-

Solve

$$P^2 + \left[x + y - \frac{2y}{x} \right] P + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0$$

$$P = \frac{y}{x} - y \text{ (or) } \frac{y}{x} - x$$

$$\frac{dy}{y} = \left[\frac{1}{x} - 1 \right] dx \text{ (or) } \frac{dy}{dx} - \frac{y}{x} = -x$$

The first equation gives

$$\text{Log } \frac{y}{x} = -x + \log c$$

$$\therefore y = c x e^{-x} \text{ ----- (1)}$$

The second equation is linear in y.

Hence the solution is

$$\frac{y}{x} = -x + c ; \text{ (i.e.) } y + x^2 - cx = 0 \text{ ----- (2)}$$

The general solution is $(y - cx e^{-x})(y + x^2 - cx) = 0$

Type : B

Let the differential equation (1) be put in the form $f(x, y, p) = 0$. When it cannot be resolved into rational linear factors as in Type A, it may be either solved for y (or) x.

Equation solvable for y :-

$$f(x, y, p) = 0 \text{ can be put in the form } y = F(x, p) \text{ ----- (1)}$$

Differentiating with respect to x,

$$\phi P = \phi \left[x, p \frac{dp}{dx} \right]$$

This being an equation in the two variables P and x can be integrated by any of the foregoing method. Hence we obtain $\psi(x, p, c) = 0$.

Eliminating 'P' between (1) and (2) the solution is got.

Equation solvable for x.

Let $f(x, y, p) = 0$ be in this case put in the form $x = F(y, p)$ ----- (1)

Differentiating with respect to y.

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Integrating leads to $\psi(y, p, c) = 0$ ----- (2)

Eliminating 'p' between (1) and (2) the solution of (1) is got.

Examples :

1) Solve $xp^2 - 2yp + x = 0$

Solving for y,

$$y = \frac{x(p^2 + 1)}{2p}$$

Differentiating with respect to x,

$$\frac{p^2 - 1}{p} = \frac{dp}{dx} \cdot \frac{p^2 - 1}{p^2} \cdot x$$

$$\therefore \frac{dx}{x} = \frac{dp}{p}$$

Integrating ; $p = cx$

We have two equation.

$$xp^2 - 2yp + x = 0.$$

$$\Rightarrow p = cx$$

$$x c^2 x^2 - 2y(cx) + x = 0$$

$$c^2 x^3 - 2cxy + x = 0.$$

$$c^2 x^2 + 1 = 2cy.$$

2. Solve

$$x = y^2 + \log p \text{ ----- (1)}$$

Differentiating with respect to 'y'

$$\frac{dx}{dy} = 2y + \frac{1}{p} \cdot \frac{dp}{dy}$$

Since $\frac{dy}{dx} = p$

$$\frac{1}{p} = 2y + \frac{1}{p} \cdot \frac{dp}{dy}$$

$$\frac{dp}{dy} + 2py = 1. \text{ This is linear in } p$$

and hence

$$pe^{y^2} = \int e^{y^2} dy + c \quad (2)$$

Eliminate of 'p' between (1) and (2) gives the solution.

Particular case :-

There are two special cases of equations solvable for y.

6.3 Clairaut's form :-

The equation known as Clairaut's is of the form.

$$y = px + f(p)$$

Differentiating with respect to x,

$$\frac{dy}{dx} = p + x \cdot \frac{dp}{dx} + f(p) \cdot \frac{dp}{dx}$$

$$p = p + (x + f'(p)) \frac{dp}{dx}$$

$$(x + f'(p)) \frac{dp}{dx} = 0$$

$$\text{either } \frac{dp}{dx} = 0 \quad (\text{or}) \quad x + f'(p) = 0$$

$$\frac{dp}{dx} = 0 \text{ gives } p = c \text{ (a constant)}$$

The solution is $y = px + f(p)$

substitute $= c$.

$$y = cx + f(p)$$

$$\Rightarrow y = cx + f(c)$$

We have to replace $-p$ in Clairaut's equation by c . The other factor $x + f'(p) = 0$ taken along with (1) give, an elimination of p , a solution of (1). But this solution is not included in the general solution (2). Such a solution as this is called a singular solution.

Examples :-

1) Solve $y = (x - a)p - p^2$

This is Clairaut's equation hence the solution is $y = (x - a)c - c^2$

2) Solve $y = 2px + y^2 p^3$.

Putting $x = 2x$ and $y = y^2$, the equation transforms into $y = xp + p^3$

Where $p = \frac{dy}{dx} = py$

This is Clairaut's equation. Hence $y = cx + c^3$.

The solution is $y^2 = 2xc + c^3$

We have an extended form of Clairaut's equation of the type.

$$y = x f(p) + \phi(p) \quad \text{----- (1)}$$

Differentiating with respect to x -----

$$\frac{dy}{dx} = f(p) + x f'(p) \cdot \frac{dp}{dx} + \phi'(p) \cdot \frac{dp}{dx}$$

$$p = f(p) + (x f'(p) + \phi'(p)) \frac{dp}{dx}$$

$$\frac{dx}{dp} + \frac{x f'(p)}{f(p) - p} = \frac{\phi'(p)}{p - f(p)}$$

This is linear in x and hence gives

$$F(x, p, c) = 0.$$

The elimination of p between this equation and (1) gives the solution of (1).

Example :-

Solve $y = xp + x(1+p^2)^{1/2}$ ----- (2)

Differentiating with respect to 'x'

$$\frac{dy}{dx} = p + (1+p^2)^{1/2} + x \cdot \frac{1}{2} (1+p^2)^{-1/2} \cdot \frac{dp}{dx}$$

$$p = p + (1+p^2)^{1/2} + \frac{dp}{dx} \cdot \frac{x}{2} \cdot \frac{1}{\sqrt{1+p^2}}$$

$$\frac{dp}{dx} \cdot \frac{x}{2} \cdot \frac{1}{\sqrt{1+p^2}} = - (1+p^2)^{1/2}$$

$$dp \frac{\sqrt{1+p^2} + p}{(1+p^2)} + \frac{dx}{x} = 0$$

Integrating,

$$\log (p \sqrt{1+p^2} + 1+p^2) \log x = \log c$$

$$(p (\sqrt{1+p^2}) + 1+p^2) x = c \text{ ----- (2)}$$

Eliminating 'p' between (1) and (2). the solution is got.

6.4. Equation the do not contain 'x' explicitly

Suppose an equation is of the form

$$f(y, p) = 0 \text{ ----- (1)}$$

If this is solvable for p, then $p = \phi (y)$ and hence is immediately integrable.

If (1) is solvable for y. So that $y = \phi (p)$. Then the method of equation solvable for y is applied.

Equation they do not contain 'y' explicitly.

Let the equation be $f(x, p) = 0$. ----- (1)

If this is solvable for p. So that $p = \phi (x)$, it is directly integrable.

If (1) is solvable for x, the method of equations solvable for x is applied.

Equations homogeneous in x and y :-

Let the equation be $f (\frac{y}{x}, p) = 0$ ----- (1)

If this is solvable for p, then $p = F (\frac{y}{x})$ and is immediately integrable.

If (1) is solvable for $\frac{y}{x}$, so that $y = xF(p)$, the we proceed as in

clairaut's equation of the type $y = x f(p) + \phi (p)$.

Differentiate with respect to x, we have

$$p = F(p) + x F'(p) \frac{dp}{dx}$$

$$\frac{dx}{x} = \frac{F'(p) dp}{p - F(p)}$$

This is integrable and the eliminate of p between this equation and (1) is the required solution

Example :-

1) Solve $x^2 = 1 + p^2$ ----- (1)

$x = \pm \sqrt{1 + p^2}$. Here y is explicitly absent.

Differentiate with respect to "y".

$$\frac{1}{p} = \frac{p}{\sqrt{1 + p^2}} \cdot \frac{dp}{dy}$$

$$dy = \frac{p^2}{\sqrt{1 + p^2}} dp$$

$$\text{Hence } y + c = \int \frac{p^2}{\sqrt{1 + p^2}} dp.$$

$$= (p \sqrt{1 + p^2} - \sinh^{-1} p) \text{ ----- (2)}$$

Eliminate ' p ' between (1) and (2), the solution is got.

2) Solve

$$xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$$

This is homogeneous in x and y and solvable for p .

$$p = \frac{2y}{x} \text{ (or) } - \frac{3x}{y}$$

$$\frac{dy}{y} = \frac{2dx}{x} \text{ (or) } ydy + 3xdx = 0$$

$$y = cx^2 \text{ (or) } y^2 + 3x^2 = c.$$

The solution is $(y - cx^2)(y^2 + 3x^2 - c) = 0$.

3. Solve

$$(xp - y)^2 = a(1 + p^2) \phi(x^2 + y^2)$$

$$(xdy - ydx)^2 = a(dx^2 + dy^2) \phi(x^2 + y^2)$$

Changing to polar coordinate.

$xdy - ydx = r^2 d\theta$. [equality elements of area in Cartesian and polar coordinates] and $dx^2 + dy^2 = ds^2 = dr^2 + r^2 d\theta^2$.

Where ds is the element of arc.

\therefore The equation transforms into

$$r^4 (d\theta)^2 = [dr^2 + r^2 (d\theta)^2] a \phi(r^2)$$

$$\therefore d\theta = \frac{dr}{r} \left\{ \frac{a \phi(r^2)}{r^2 - a \phi(r^2)} \right\}^{1/2}$$

$$\therefore \theta + c = \int \frac{dr}{r} \left\{ \frac{a \phi(r^2)}{r^2 - a \phi(r^2)} \right\}^{1/2}$$

Exercises :-

$$1) y = px + \frac{ap}{(1+p^2)^{1/2}}$$

$$2) y \cdot \frac{dp}{dx} + p^2 = 1$$

$$3) p^2 - 5p + 6 = 0$$

$$4) y^2 = (1+p^2)$$

$$5) xp(3y^2 - ax) = y(2y^2 - ax)$$

[Hint : Put $y = ux$; a linear equation is got]

Homogeneous linear equation :-

Consider a differential equation of the form

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + p_{n-1} x \frac{dy}{dx} + p_n y = x \dots \quad (1)$$

Where p_1, p_2, \dots, p_n are constants and x is a function of x . This equation is called a homogeneous linear equation.

Equation (1) can be transformed to a linear equation with constant coefficients by the substitution

$$z = \log x \text{ (i.e.) } x = e^z$$

We have $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$

$$\therefore x \cdot \frac{dy}{dx} = \frac{dy}{dz}$$

$$\therefore x Dy = \theta y \text{ where } D = \frac{d}{dx} \text{ and } \theta = \frac{d}{dz}$$

Differentiating w.r. to x again we get

$$\frac{d^2y}{dx^2} = \frac{1}{x^2} \left(\frac{d^2y}{dz^2} - \frac{dy}{dz} \right)$$

$$\begin{aligned} \therefore x^2 D^2 y &= (\theta^2 - \theta) y. \\ &= \theta(\theta - 1) y. \end{aligned}$$

Similarly $x^3 D^3 y = \theta(\theta - 1)(\theta - 2) y$

$$x^n D^n y = \theta(\theta - 1)(\theta - 2) \dots (\theta - n + 1) y$$

\therefore (1) is transformed to

$$[\theta(\theta - 1) \dots (\theta - n + 1) + p_1(\theta - 1) \dots (\theta - n + 2) + \dots + p_{n-1}\theta + p_n] y = z \quad (2)$$

Where z is a function of ' z ' obtained from x by putting $x = e^z$. hence (2) is a linear equation with constant coefficients and the complementary function can be found by the methods described in finding complementary functions.

We now give a method of finding particular integral for (2). This method can be used for problems whose principal integral (PI) is difficult to evaluated by the methods described in finding particular integrals.

Consider $\frac{1}{f(\theta)} x$

We first express $\frac{1}{f(\theta)}$ into partial fractions.

Consequently $\frac{1}{f(\theta)} x$ can be expressed as

$$\left(\frac{a_1}{\theta - \alpha_1} + \frac{a_2}{\theta - \alpha_2} + \dots + \frac{a_n}{\theta - \alpha_n} \right) x \text{ where } a_i \text{ s are constants.}$$

Hence the problem of finding the P. I ultimately depends on the evaluation

of $\frac{1}{\theta - \alpha} x$.

Let $\frac{1}{\theta - \alpha} X = Y$. Hence $x \frac{dy}{dx} = X$

This is a linear differential equation and its solution is given by

$$y x^{-\alpha} = \int x^{-\alpha-1} x dx.$$

$$\therefore y = x^{\alpha} \int x^{-\alpha-1} x dx$$

$$\therefore \frac{1}{\theta - \alpha} x = x^{\alpha} \int x^{-\alpha-1} x dx$$

Solved problems :-

1) Solve $\frac{x^2 d^2 y}{dx^2} - 3x \frac{dy}{dx} - 5y = \sin(\log x)$

Solution

Put $z = \log x$ and $\theta = \frac{d}{dz}$

The given equation reduces to

$$(\theta(\theta - 1) - 3\theta - 5) y = \sin z.$$

(i.e.) $(\theta^2 - 4\theta - 5) y = \sin z.$

Auxiliary equation is $m^2 - 4m - 5 = 0$

$$\therefore (m - 5)(m + 1) = 0.$$

$$\therefore c - F = c_1 e^{5x} + c_2 e^{-x}$$

$$= c_1 e^{5 \log x} + c_2 e^{-\log x}$$

$$= c_1 x^5 + c_2 x^{-1}$$

$$PI = \left[\frac{1}{\theta^2 - 4\theta - 5} \right] \sin z$$

$$= \left[\frac{1}{-1 - 4\theta - 5} \right] \sin z$$

$$= \left[\frac{1}{-2(2\theta+3)} \right] \sin z$$

$$= \left[\frac{2\theta-3}{-2(4\theta^2-9)} \right] \sin z$$

$$= \left[\frac{2\theta-3}{26} \right] \sin z$$

$$= \frac{2\cos z - 3 \sin z}{26}$$

$$= \frac{1}{13} \cos z - \frac{3}{26} \sin z$$

$$= \frac{1}{13} \cos (\log x) - \frac{3}{26} \sin (\log x)$$

\therefore The solution is $y = \text{c.F.} + \text{P.I.}$

2) Solve

$$x^2 y^{11} - xy^1 + 4y = \cos (\log x) + x \sin (\log x).$$

Pf :-

$$\text{Put } z = \log x \text{ and } \theta = \frac{d}{dz}$$

\therefore The given equation reduces to

$$[\theta(\theta-1) - \theta + 4] y = \cos z + e^z \sin z$$

The auxiliary equation $m^2 - 2m + 4 = 0$.

$$\therefore m = 1 \pm i\sqrt{3}$$

$$\text{Hence C.F.} = e^z (c_1 \cos \sqrt{3} z + c_2 \sin \sqrt{3} z)$$

$$= x [c_1 \cos (\sqrt{3} \log x) + c_2 \sin (\sqrt{3} \log x)]$$

$$\text{P.I.} = \left[\frac{1}{\theta^2 - 2\theta + 4} \right] \cos z + \left[\frac{1}{\theta^2 - 2\theta + 4} \right] e^z \sin z$$

$$= \frac{1}{3-2\theta} \cos z + e^z \left[\frac{1}{(\theta+1)^2 - 2(\theta+1) + 4} \right] \sin z$$

$$= \frac{1}{3-2\theta} \cos z + e^z \left[\frac{1}{\theta^2 + 3} \right] \sin z$$

$$\begin{aligned}
&= \frac{3 + 2\theta}{9 - 4\theta^2} \cos z + \frac{e^z \sin z}{(-1 + 3)} \\
&= \left(\frac{3 + 2\theta}{13} \right) \cos z + \frac{1}{2} e^z \sin z \\
&= \frac{1}{13} (3 \cos z - 2 \sin z) + \frac{1}{2} e^z \sin z \\
&= \frac{1}{13} (3 \cos (\log x) - 2 \sin (\log x)) + \frac{1}{2} x \sin (\log x)
\end{aligned}$$

∴ The solution is $y = \text{c.F.} + \text{P.I}$

3) Solve

$$x^2 y'' + 4xy' + 2y = e^x$$

$$\text{Put } z = \log x ; \theta = \frac{d}{dz} = x \frac{d}{dx}$$

The given equation reduces to $(\theta(\theta - 1) + 4\theta + 2)y = e^x$

$$(\text{i.e.}) (\theta^2 + 3\theta + 2)y = e^x$$

The auxiliary equation is $m^2 + 3m + 2 = 0$.

$$\therefore m = -1, -2.$$

$$\text{Hence c.F.} = c_1 e^{-z} + c_2 e^{-2z}$$

$$= c_1 x^{-1} + c_2 x^{-2} \quad (\text{since } z = \log x)$$

$$\text{P.I} = \frac{1}{(\theta + 1)(\theta + 2)} e^x$$

$$= \left(\frac{1}{(\theta + 1)} - \frac{1}{\theta + 2} \right) e^x$$

$$= x^{-1} \int e^x dx - x^{-2} \int x e^x dx$$

$$= x^{-1} e^x - x^{-2} (x e^x - e^x)$$

$$= x^{-2} e^x.$$

∴ The solution is $y = c_1 x^{-1} + c_2 x^{-2} + x^{-2} e_x$.

(4) Solve

$$(2x+1)^2 y^{11} - 2(2x+1)y^1 - 12y = 6x$$

Put $2x + 1 = z$. Hence $\frac{dy}{dx} = 2$

$$x = \frac{z-1}{2}$$

$$y^1 = \frac{dy}{dz} \cdot \frac{dz}{dx} = 2 \frac{dy}{dz} \text{ and } y^{11} = z \frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$$

$$y^{11} = 4 \cdot \frac{d^2y}{dz^2}$$

Hence the given equation reduces to a linear homogeneous equation.

$$4Z^2 \frac{d^2y}{dz^2} - 4Z \frac{dy}{dz} - 12y = 6 \frac{z-1}{2} \dots\dots\dots(1)$$

Now put $u = \log Z$ and $q = \frac{d}{du}$ and hence the equation reduces to

$$(4\theta(\theta-1) - 4\theta - 12)y = 3(e^u - 1)$$

$$\text{i.e. } (\theta^2 - 2\theta - 3)y = \frac{3}{4}(e^u - 1)$$

The auxiliary equation is $m^2 - 2m - 3 = 0$.

$$(m-3)(m+1) = 0 \text{ Hence } m = 3, -1$$

$$\begin{aligned} \text{C.F.} &= c_1 e^{3u} + c_2 e^{-u} \\ &= c_1 e^{3 \log z} + c_2 e^{-\log z} \\ &= c_1 z^3 + c_2 z^{-1} \\ &= c_1 (2x+1)^3 + c_2 (2x+1)^{-1} \end{aligned}$$

$$\text{PI} = \frac{3}{4} \left[\frac{1}{\theta^2 - 2\theta - 3} \right] (e^u - 1)$$

$$= \frac{3}{4} \left[\frac{e^u}{-4} + \frac{1}{3} \right]$$

$$= \frac{3}{4} \left[\frac{-z}{4} + \frac{1}{3} \right]$$

$$= \frac{3}{4} \left(\frac{2x+1}{-4} + \frac{1}{3} \right)$$

$$= \frac{-3x}{8} + \frac{1}{16}$$

Hence the solution is $y = \text{C.F.} + \text{P.I.}$

Exercises :

Solve the following differential equation.

1. $x^2 y'' + xy' - y = 0$

2. $x^2 y'' - 2xy' - 4y = x^4$

3. $x^2 y'' + 2xy' - 20y = (x+1)^2$

4. $x^2 y'' + 4xy' + 2y = \text{Log } x$

5. $x^2 y'' - 3xy' - 5y = \text{Sin Log } x.$

6.5 Linear equations with constant coefficients

An n^{th} order linear equation is of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = x$$

Where P_1, \dots, P_n are functions of x .

If however P_1, \dots, P_n are constants. The above equation is called a linear equation of the n^{th} order with constant coefficient.

Linear equation of the second order with constant coefficients :

A linear equation of this type is of the form

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = x$$

Where a, b, c are constants and x is a function of x .

First consider the equation where $x=0$.

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \rightarrow (2)$$

Suppose $y=y_2$ and $y=y_1$ are two solutions of (2).

$$\therefore a \frac{d^2 y_1}{dx^2} + b \frac{dy_1}{dx} + cy_1 = 0$$

$$a \frac{d^2 y_2}{dx^2} + b \frac{dy_2}{dx} + cy_2 = 0$$

$$(i.e) \quad a \frac{d^2}{dx^2} (y_1 + y_2) + b \frac{d}{dx} (y_1 + y_2) + c (y_1 + y_2) = 0$$

$\therefore y_1 + y_2$ is also a solution of (2)

Also if y_1 and y_2 are solution then $c_1 y_1$ and $c_2 y_2$ are solutions for any arbitrary constants c_1 and c_2 .

$\therefore c_1 y_1 + c_2 y_2$ is also a solution of (2)

$$\text{For } a \frac{d^2}{dx^2} (c_1 y_1 + c_2 y_2) + b \frac{d}{dx} (c_1 y_1 + c_2 y_2) + c (c_1 y_1 + c_2 y_2)$$

$$= c_1 a \frac{d^2 y_1}{dx^2} + b \frac{dy_1}{dx} + c_1 y_1 + c_2 a \frac{d^2 y_2}{dx^2} + b \frac{dy_2}{dx} + c y_2$$

$$= c_1 0 + c_2 0 = 0 \rightarrow (3)$$

Let $y = V$ be any particular solution of (1)

$$\therefore a \frac{d^2 V}{dx^2} + b \frac{dV}{dx} + cV = x \rightarrow (4)$$

Adding (3) and (4)

$$a \frac{d^2}{dx^2} (c_1 y_1 + c_2 y_2 + V) + b \frac{d}{dx} (c_1 y_1 + c_2 y_2 + V) + c (c_1 y_1 + c_2 y_2 + V) = 0$$

Hence $c_1 y_1 + c_2 y_2 + V$ is a solution of (1)

$c_1 y_1 + c_2 y_2 + V$ is called the complete solution of (1)

The complete solution consists of two parts.

(i) $c_1 y_1 + c_2 y_2 +$ which is the general solution of the equation

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0 \text{ and}$$

$$(ii) \text{ A Particular solution } V \text{ of } a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = X \text{ and}$$

The first part is called the complementary function (C.F) and the second part is called the particular integral. (P.I)

The symbol D :

The symbols Dy , D^2y , D^3y are used to denote respectively the first, second, third..... derivatives of y w.r. to x .

In this notation the equation.

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = x \text{ becomes}$$

$$aD^2y + bDy + cy = x \text{ which can be written as } (aD^2 + bD + c) y = x.$$

To obtain the complementary Function :

Let $y = e^{mx}$ for some value of m be a trial solution of $(aD^2 + bD + c) y = 0$.

$$\therefore (aD^2 + bD + c) e^{mx} = 0 \rightarrow (A)$$

$$(am^2 + bm + c) e^{mx} = 0$$

$$\therefore am^2 + bm + c = 0 \quad (\because e^{mx} \neq 0) \rightarrow (B)$$

The equation $am^2 + bm + c = 0$ ($\because e^{mx} \neq 0$)..... is called the Auxillary equation (A.E).

Let m_1 and m_2 be the roots of the A.E. Then e^{m_1x} and e^{m_2x} are two solutions of (A). Since A general solution of (A) is given by $c_1 e^{m_1x} + c_2 e^{m_2x}$ where c_1 and c_2 are arbitrary constants.

There are three cases :

Case : 1

If the two roots m_1 and m_2 of the A.E. are real and distinct then the solution is $Y = c_1 e^{m_1x} + c_2 e^{m_2x}$

Case : 2

Let the roots be equal. In this case the given equation is of the form.

$$a (D-m_1) (D-m_1) y = 0$$

$$(D-m_1) (D-m_1) y = 0$$

$$\text{Put } (D-m_1) y = Z.$$

$$\therefore (D-m_1) Z = 0.$$

$$\Rightarrow \frac{dz}{dx} - m_1 z = 0$$

$$\frac{dz}{dx} - m_1 z = 0$$

$$\Rightarrow \log Z = m_1 x + k.$$

$$\Rightarrow Z = e^{m_1 x} + k.$$

$$(D-m_1) y = c_1 e^{m_1 x}$$

$$\Rightarrow \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$$

$$\Rightarrow y e^{-m_1 x} = \int c_1 e^{m_1 x} e^{-m_1 x} dx + c_2.$$

$$\Rightarrow y e^{-m_1 x} = c_1 x + c_2.$$

$$\Rightarrow y = e^{m_1 x} (c_1 x + c_2)$$

Thus when the roots of the A-E are equal the solution is

$$Y = e^{m_1 x} (c_1 + c_2)$$

Case : 3

Suppose the roots of A.E are conjugate complex, say $d+i\mathcal{B}$ and $d-i\mathcal{B}$.

Then the solution is

$$Y = C_1 e^{(\alpha+i\mathcal{B})x} + C_2 e^{(\alpha-i\mathcal{B})x}$$

$$= e^{\alpha x} (C_1 e^{i\mathcal{B}x} + C_2 e^{-i\mathcal{B}x})$$

$$= e^{\alpha x} C_1 (\cos \mathcal{B}x + i \sin \mathcal{B}x) + C_2 (\cos \mathcal{B}x + i \sin \mathcal{B}x)$$

$$= e^{\alpha x} \cos \mathcal{B}x (C_1 + C_2) + \sin \mathcal{B}x (iC_1 - iC_2)$$

$$= e^{\alpha x} (A \cos \mathcal{B}x + B \sin \mathcal{B}x)$$

Working Rule :

For solving $(aD^2 + bD + c) y = 0$

1. Write the A.E which is quadratic.
2. Find the roots m_1 and m_2 of A.E.
3. The solution is

i) $Y = Ae^{m_1x} + Be^{m_2x}$ if m_1 and m_2 are real and distinct.

ii) $Y = e^{m_1x} (Ax + B)$ if $m_1 = m_2$.

iii) $Y = e^{\alpha x} (A \cos \beta x + B \sin \beta x)$ if the roots are $\alpha + i\beta$ and $\alpha - i\beta$.

Where A and B are constants.

To obtain the particular integral :

Consider the equation $(aD^2 + bD + c) y = x$.

Where x is a function of x.

(i.e) $f(D) y = x$

$y = x/f(D)$ is called the particular

integral (PI) of $f(D) y = x$.

$$(i.e.) PI = \frac{1}{aD^2 + bD + c}$$

4. a) x is of the form $e^{\alpha x}$ where $f(\alpha) \neq 0$.

$$D(e^{\alpha x}) = \alpha e^{\alpha x}$$

$$D^2(e^{\alpha x}) = \alpha^2 e^{\alpha x} \text{ etc}$$

$$\text{generally } D^n(e^{\alpha x}) = \alpha^n e^{\alpha x}.$$

$$\text{More generally } f(D)(e^{\alpha x}) = f(\alpha) e^{\alpha x}$$

$$\therefore \frac{1}{f(D)}(f(D) e^{\alpha x}) = \frac{1}{f(\alpha)}(f(\alpha) e^{\alpha x})$$

$$(i.e) e^{\alpha x} = f(\alpha) \frac{1}{f(D)} e^{\alpha x}$$

$$\frac{1}{f(x)} e^{\alpha x} = \frac{1}{f(D)} e^{\alpha x}$$

$$\therefore \frac{1}{f(D)} e^{\alpha x} = \frac{1}{f(\alpha)} e^{\alpha x} \text{ where } f(\alpha) \neq 0$$

Example :

$$\text{Solve } (D^2 - 5D + 6) y = 2e^{4x}.$$

To find the C.F

$$\text{Solve } (D^2 - 5D + 6) y = 0.$$

The Auxiliary equation is $m^2 - 5m + 6 = 0$

$$\text{(i.e.) } (m-2)(m-3) = 0.$$

$$\text{(i.e.) } m = 2 \text{ and } 3$$

$$\therefore \text{The C.F} = Ae^{2x} + Be^{3x}.$$

To find the P.I.

$$\text{P.I} = \frac{2e^{4x}}{D^2 - 5D + 6}.$$

$$= \frac{2e^{4x}}{2} \text{ putting } D = 4.$$

$$= e^{4x}.$$

\therefore The solution is $y = \text{C.F} + \text{P.I.}$

$$\text{(i.e.) } y = Ae^{2x} + Be^{3x} + e^{4x}$$

Example :

$$\text{Solve } (D^2 + 2D + 5) y = \sin hx.$$

$$(D^2 + 2D + 5) y = \frac{e^x - e^{-x}}{2}$$

To find the C.F.

$$\text{Solve } (D^2 + 2D + 5) y = 0$$

The A.E is $m^2 + 2m + 5 = 0.$

To find the C.F

$$\text{Solve } (D^2 + 2D + 5) y = 0$$

The A.E is $m^2 + 2m + 5 = 0$

Working Rule : $(i.e) m = \frac{-2 \pm \sqrt{4 - 20}}{2}$

$$= -1 \pm 2i.$$

\therefore The C.F = $e^{-x} (A \cos 2x + B \sin 2x)$

$$P.I_1 \text{ corresponding to } \frac{e^x}{2} = \frac{1}{D^2 + 2D + 5} \left(\frac{e^x}{2} \right)$$

$$= \frac{1}{16} e^x$$

$$P.I_2 \text{ corresponding to } \frac{-e^{-x}}{2} = \frac{1}{D^2 + 2D + 5} \left(\frac{-e^{-x}}{2} \right)$$

$$= \frac{-e^{-x}}{2}$$

\therefore The Solution is $Y = C.F + P.I_1 + P.I_2$.

$$(i.e.) Y = e^{-x} (A \cos 2x + B \sin 2x) + \frac{-e^{-x}}{16} - \frac{e^{-x}}{8}$$

Exercises :

1. $(D^2 + 5D + 6) y = e^x$.

2. $(D^2 + 4D + 6) y = 5e^{-2x}$.

3. $(D^2 + 2D + 1) y = 2e^{3x}$.

4. $(D^2 - 3D + 2) y = 2e^{3x}$ given that $y=0$ when $x=0$ and $x=\log^2$.

X is of the form $e^{\alpha x}$ where $f(x) = 0$:

Since $f(\alpha) = 0$; α is a root of $aD^2 + bD + c = 0$.

Let m be the other root.

$$\therefore aD^2 + bD + c = a (D - \alpha) (D - m)$$

\therefore The differential equation becomes

$$a (D - \alpha) (D - m) y = e^{\alpha x}.$$

$$\therefore \text{P.I. is } y = \frac{e^{\alpha x}}{a(D-\alpha)(D-m)}$$

$$= \frac{1}{a(\alpha-m)} \frac{e^{\alpha x}}{(D-\alpha)}$$

$$\text{Now } = \frac{e^{\alpha x}}{(D-\alpha)} = z \text{ gives}$$

$$\text{Now } = \frac{e^{\alpha x}}{(D-\alpha)} = z \text{ gives}$$

$$(D-\alpha) Z = e^{\alpha x}$$

$$\text{(i.e.) } \frac{dZ}{dx} - \alpha Z = e^{\alpha x} \text{ which is linear in } Z$$

$$\text{(i.e.) } Ze^{-\alpha x} = \int e^{\alpha x} e^{-\alpha x} dx$$

$$= x.$$

$$\therefore Z = xe^{\alpha x}.$$

$$\therefore \frac{e^{\alpha x}}{D-\alpha} = xe^{\alpha x}.$$

$$\text{Similarly } \frac{1}{(D-\alpha)^2} e^{\alpha x} = \frac{1}{D-\alpha} (xe^{\alpha x}) = V \text{ say}$$

$$\frac{dv}{dx} - \alpha v = xe^{\alpha x} \text{ which is linear in } V.$$

$$\therefore V \cdot e^{-\alpha x} = \int x e^{\alpha x} e^{-\alpha x} dx$$

$$V \cdot e^{-\alpha x} = \frac{x^2}{2}$$

$$\therefore V = \frac{x^2}{2} e^{\alpha x}$$

$$\therefore \frac{1}{(D-\alpha)^2} e^{\alpha x} = \frac{x^2}{2!} e^{\alpha x}$$

In general we can prove that

$$\frac{1}{(D-\alpha)^m} e^{\alpha x} = \frac{x^m}{m!} e^{\alpha x} : m \in \mathbb{N}$$

Example :

Solve $(D^2 + 6D + 9) y = e^{-3x}$.

To find the C.F. Solve $(D^2 + 6D + 9) y = 0$.

The A.E is $m^2 + 6m + 9 = 0$

$(m + 3)(m + 3) = 0$.

$m = -3$ and $m = -3$.

\therefore The c.F = $e^{-3x} (Ax + B)$

To find the P.I.

$$\begin{aligned} \text{P. I} &= \frac{e^{-3x}}{D^2 + 6D + 9} \\ &= \frac{e^{-3x}}{(D + 3)^2} = \frac{x^2}{2} e^{-3x} \end{aligned}$$

The solution is $Y = \text{C.F} + \text{PI}$.

$$\begin{aligned} \text{(i.e.) } y &= e^{-3x} (Ax + B) + \frac{x^2}{2} e^{-3x} \\ &= e^{-3x} \left(\frac{x^2}{2} + Ax + B \right) \end{aligned}$$

Exercises :

1. $(D^2 + 6D + 8) y = e^{-2x}$

2. $(D^2 + D - 6) y = \text{Cosh } 3x$

3. $(3D^2 + D - 14) y = 13e^{2x}$

X is of the form $\text{Sin } \alpha x$ or $\text{Cos } \alpha x$ where $f(-\alpha)^2 \neq 0$:

$D (\text{Sin } \alpha x) = \alpha \text{Cos } \alpha x$.

$D^2 (\text{Sin } \alpha x) = -\alpha^2 \text{Sin } \alpha x$.

$D^2 (\text{Cos } \alpha x) = -\alpha^2 \text{Cos } (\alpha x)$.

$$D^4 (\sin \alpha x) = \alpha^4 \sin \alpha x.$$

$$(i.e.) (D^2)^2 (\sin \alpha x) = (-\alpha^2)^2 \sin \alpha x.$$

$$\text{Similarly } (D^2)^3 (\sin \alpha x) = (-\alpha^2)^3 \sin \alpha x.$$

$$\text{In general } (D^2)^n (\sin \alpha x) = (-\alpha^2)^n \sin \alpha x.$$

$$\text{More general } f(D^2) \sin \alpha x = f(-\alpha^2) \sin \alpha x.$$

$$\text{Now } \sin \alpha x = \frac{1}{f(D^2)} f(D^2) \sin \alpha x.$$

$$= \frac{1}{f(D^2)} f(-\alpha^2) \sin \alpha x.$$

$$(i.e.) \frac{1}{f(D^2)} \sin \alpha x = \frac{1}{f(-\alpha^2)} \sin \alpha x \text{ Provided } f(-\alpha^2) \neq 0.$$

$$\text{Similarly } \frac{1}{f(D^2)} \cos \alpha x = \frac{1}{f(-\alpha^2)} \cos \alpha x$$

Rule :

$$\text{In } \frac{1}{f(D^2)} \sin \alpha x \text{ replace } D^2 \text{ by } -\alpha^2.$$

$$\text{The equation reduces to } \frac{1}{\ell D + m} \sin \alpha x.$$

$$(i.e.) \frac{LD - m}{L^2 D^2 - m^2} \sin \alpha x.$$

$$\text{Again replace } D^2 \text{ by } -\alpha^2.$$

$$\therefore \text{ We have } K (LD - m) \sin \alpha x; K \text{ being a constant.}$$

$$(i.e.) K (\ell \alpha \cos \alpha x - m \sin \alpha x)$$

Example :

$$\text{Solve } (D^2 + D + 1)y = \sin^2 x$$

$$\text{The A.E. is } m^2 + m + 1 = 0$$

$$\therefore m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\therefore \text{C.F} = e^{-x/2} \left(A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x \right)$$

$$\text{P.I} = \frac{1}{D^2 + D + 1} \sin 2x$$

$$= \frac{\sin 2x}{D - 3} \text{ replacing } D^2 \text{ by } -4.$$

$$= \frac{(D + 3) \sin 2x}{D^2 - 9}$$

$$= \frac{-1}{13} (D + 3) \sin 2x$$

$$= \frac{-1}{13} (2 \cos 2x + 3 \sin 2x)$$

\therefore The solution is

$$y = e^{-x/2} \left(A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x \right) - \frac{1}{13} (2 \cos 2x + 3 \sin 2x)$$

Exercises : Solve :

1. $(D^2 - 3D + 2) y = \sin 3x$
2. $(D^2 - 4D + 3) y = \sin 3x \cos 2x.$
3. $(D^2 - 4) y = \sin^2 x.$

X is of the form $\sin \alpha x$ (or) $\cos \alpha x$ where $f(-\alpha^2) = 0$.

If $f(-\alpha^2) = 0$ we adopt the following method

$$\text{P.I.} = \frac{1}{aD^2 + bD + c} \cos \alpha x.$$

Exercises :

$$\text{Solve the following} = \text{Real part of } \frac{1}{a(D+i\alpha)(D-i\alpha)} e^{i\alpha x}$$

$$= \text{Real part of } \frac{1}{a(2i\alpha)} x e^{i\alpha x}$$

$$= \text{Real part of } \frac{1}{i 2\alpha} x (\cos \alpha x + i \sin \alpha x)$$

$$\therefore \frac{1}{2D^2+bD+c} \cos \alpha x = \frac{x}{2\alpha} \sin \alpha x.$$

Similarly for $\frac{1}{2D^2+bD+c} \sin \alpha x$. We take the imaginary part.

Exercises :

$$1. (D^2+4) y = \sin^2 x.$$

$$2. (D^2+1) y = 2 \cos^2 x/2.$$

$$3. (D^2+9) y = \cos^3 x.$$

X is of the form x^m :

$$P.I = \frac{1}{2D^2+bD+c} x^m$$

$$= \frac{1}{c} \frac{x^m}{\left(1 + \frac{b}{c}D + \frac{a}{c}D^2\right)}$$

$$= \frac{1}{c} \left(1 + \frac{b}{c}D + \frac{a}{c}D^2\right)^{-1} x^m.$$

$$= \frac{1}{c} \left(1 + \frac{b}{c}D + \frac{a}{c}D^2\right) + \frac{b}{c}D + \frac{a}{c}D^2 \dots x^m.$$

$$= \frac{1}{c} [1 + K_1 D + K_2 D^2 + \dots] x^m.$$

$$D(x^m) = m x^{m-1}.$$

$$D^2(x^m) = m(m-1) x^{m-2}.$$

$$D^3(x^m) = m(m-1)(m-2) x^{m-3} \text{ etc.}$$

$$D^m(x^m) = m!$$

using the above results we get the solution.

Example :

$$\text{Solve } (D^2 + 3D + 2) y = x^2.$$

$$\text{The A.E is } m^2 + 3m + 2 = 0$$

$$(\text{i.e.}) (m+2)(m+1) = 0.$$

$$\therefore m = -1 \text{ and } m = -2.$$

$$\therefore \text{C.F is } Ae^{-x} + Be^{-2x}$$

$$\text{P.I} = \frac{1}{D^2 + 3D + 2} x^2$$

$$= \frac{1}{2 \left(1 + \frac{3}{2} D + \frac{1}{2} D^2 \right)} x^2$$

$$= \frac{1}{2} \left[1 - \frac{3}{2} D + \frac{1}{2} D^2 \right] + \left[\frac{3}{2} D + \frac{1}{2} D^2 \right]^2 + \dots x^2$$

$$= \frac{1}{2} \left[x^2 - \frac{3}{2} D(x^2) - \frac{1}{2} D^2(x^2) + \frac{9}{4} D^2(x^2) \right]$$

$$= \frac{1}{2} \left[x^2 - \frac{3}{2} \cdot 2x - \frac{1}{2} \cdot 2 + \frac{9}{4} \cdot 2 \right]$$

$$= \frac{1}{2} \left[x^2 - 3x + \frac{7}{2} \right]$$

$$\text{The solution is } y = Ae^{-x} + Be^{-2x} + \frac{1}{2} \left[x^2 - 3x + \frac{7}{2} \right]$$

Exercises :

Solve the following equations :-

1. $(D^2 + D + 1) y = x$
2. $(D^2 + 9) y = 2x^2$.
3. $(D^2 + 3D - 4) y = x^2 - 2x$.

X is of the form $e^{ax} V$ where V is a function of x:-

$$D (e^{ax} V) = ae^{ax} V + e^{ax} DV$$

$$= e^{ax} (D + a) V$$

$$D^2 (e^{ax} V) = D(D^{ax} V)$$

$$= D(e^{ax} (D + a) v)$$

$$= ae^{ax} (D + a) v + e^{ax} (D^2 + aD) v$$

$$= e^{ax} (D^2 + 2aD + a^2) v$$

$$= e^{ax} (D + a^2) v.$$

Generally,

$$D^n (e^{ax} v) = e^{ax} (D + a)^n V$$

$$\therefore f(D) \{e^{ax} v\} = e^{ax} f(D + a) V$$

$$\therefore e^{ax} v = \frac{1}{f(D)} f(D) (e^{ax} V)$$

$$= \frac{1}{f(D)} e^{ax} f(D + a) v$$

$$\text{Let } f(D + a) V = V_1$$

$$\therefore V = \frac{1}{f(D + a)} V_1$$

$$\therefore e^{ax} \frac{1}{f(D + a)} V_1 = \frac{1}{f(D)} e^{ax} V_1$$

$$\text{Hence } \frac{1}{f(D)} e^{ax} V_1 = e^{ax} \frac{1}{f(D + a)} V_1$$

Example :

$$\text{Solve } (D^2 - 2D + 2) y = e^x \cos x.$$

$$\text{A.E is } m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$$

$$\therefore \text{ c.f } = e^x (A \cos x + B \sin x)$$

$$\text{P. I} = \frac{1}{D^2 - 2D + 2} (e^{x \cos x})$$

$$= \frac{1}{e^x} = \frac{1}{(D+1)^2 - 2(D+1) + 2} \cos x$$

Replacing D by D + 1

$$= \frac{1}{e^x} \frac{1}{D^2 + 1} \cos x$$

$$= e^x \frac{1}{(D+i)(D-i)} \text{ Real part of } e^{ix}$$

$$= e^x \text{ Real part of } \frac{e^{ix}}{(D+i)(D-i)}$$

$$= e^x \text{ Real part of } \frac{1}{2i} \left(\frac{e^{ix}}{D-i} \right)$$

$$= e^x \text{ R.P. of } \frac{1}{2i} x e^{ix}$$

$$= e^x \text{ R.P of } \frac{1}{2i} x (\cos x + i \sin x)$$

$$= \frac{e^x}{2} x \sin x.$$

∴ The solution is $y = e^x (A \cos x + B \sin x)$

$$+ \frac{e^x}{2} x \sin x$$

Example:

$$\text{Solve } (D^2 + 1) y = x^2 e^{2x} + x \cos x$$

$$\text{A.E is } m^2 + 1 = 0$$

$$m = \pm i$$

$$\therefore \text{c.F} = A \cos x + B \sin x.$$

$$\text{P.I corresponding to } x^2 e^{2x} = \frac{1}{D + i^2} x^2 e^{2x}$$

$$= e^{2x} \frac{1}{(D+2)^2 + 1} x^2 \text{ replacing } D \text{ by } D+2$$

$$= e^{2x} \frac{x^2}{D^2 + 4D + 5}$$

$$= \frac{e^{2x}}{5} \frac{x^2}{1 + \frac{4}{5}D + \frac{D^2}{5}}$$

$$= \frac{e^{2x}}{5} \frac{x^2}{\left(1 + \frac{4}{5}D + \frac{D^2}{5}\right)^{-1}}$$

$$= \frac{e^{2x}}{5} \left(1 - \frac{4}{5}D - \frac{1}{5}D^2 + \frac{16}{25}D^2\right) x^2$$

$$= \frac{e^{2x}}{5} \left(x^2 - \frac{8x}{5} - \frac{2}{5} + \frac{32}{25}\right)$$

$$= \frac{e^{2x}}{5} \left(x^2 - \frac{8x}{5} + \frac{22}{25}\right)$$

P.I corresponding to $x \cos x$

$$= \frac{1}{D^2 + 1} x \cos x$$

$$= \frac{1}{D^2+1} \text{ Real part of } x e^{ix}$$

$$= \text{Real part of } \frac{1}{D^2+1} x e^{ix}$$

$$= \text{R.P of } e^{ix} \frac{1}{(D+i)^2+1} x$$

$$= \text{R.P of } e^{ix} \frac{1}{D^2+2iD} x$$

$$= \text{R. P of } e^{ix} \frac{1}{2iD + \left(1 + \frac{D}{2i}\right)} x$$

$$= \text{R. P of } e^{ix} \frac{\left(1 + \frac{D}{2i}\right)^{-1} x}{2iD}$$

$$= \text{R.P of } e^{ix} \frac{(-i) \left(1 + \frac{D}{2i} - \frac{D^2}{4}\right)}{2D} x$$

$$= \text{R.P of } \frac{-ie^{ix}}{2} \left(\frac{1}{D} + \frac{i}{2} - \frac{D}{4} \right) x$$

$$= \text{R.P of } \frac{-ie^{ix}}{2} \left(\frac{x^2}{2} + \frac{ix}{2} - \frac{1}{4} \right) x$$

$$= \text{R.P of } \frac{-i}{2} (\text{Cos} x + i \text{sin} x) \left(\frac{x^2}{2} + \frac{ix}{2} - \frac{1}{4} \right)$$

$$= \frac{x}{4} \text{Cos } x + \frac{x^2}{4} \text{Sin} x - \frac{1}{8} \text{Sin} x$$

The solution is $y = \text{C.F} + \text{PI}_1 + \text{PI}_2$.

Exercises :

Solve the following equations :

1. $(D^2 - 2D + 4) y = e^x \text{Sin} x.$

2. $(D^2 + 2D + 5) y = x e^x.$

3. $(D^2 + 4) y = x \text{Sin } x$

4. $(D^2 - 1) y = x \text{sin} x + (1 + x^2) c^x.$

UNIT - 7

7.1 Linear equations of the second order with variable coefficients :

A linear homogeneous equation of the n^{th} order is of the form

$$X^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = 0.$$

where P_1, P_2, \dots, P_n are all constants and x is a function of x .

For solving such an equation we let

$x = e^z$ so that $z = \log x$.

$$\therefore \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{x} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx} = \frac{1}{x^2} \cdot \frac{d^2 y}{dz^2}$$

$$= \frac{1}{x^2} \cdot \frac{d^2 y}{dz^2} - \frac{1}{x^2} \cdot \frac{dy}{dz}$$

$$\therefore x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz}$$

If we use D to denote $\frac{d}{dz}$, then

$$x \cdot \frac{dy}{dx} = Dy.$$

$$x^2 \frac{d^2 y}{dx^2} = D^2 y - Dy = D(D-1)y$$

Similarly $x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$ and so on.

Method of solving homogeneous linear equation of the second order :

A homogeneous linear equation of the second order with variable coefficients is of the form

$$\frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = x \quad \dots\dots\dots(1)$$

Putting $z = \log x$. we have

$$x \cdot \frac{dy}{dx} = Dy \quad \text{and} \quad x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$\text{Where } D = \frac{d}{dz}$$

\therefore (1) becomes

$$aD(D-1)y + bDy + cy = x$$

$$\text{(i.e) } \{aD(D-1) + bD + c\} y = x \quad \dots\dots\dots(2)$$

(2) is a linear equation of the second order with constant coefficient and hence can be solved.

$$\text{To find the P.I. Let } \theta = x \frac{d}{dy}$$

The equation (1) becomes

$$\{a\theta(\theta-1) + b\theta + c\} y = x$$

$$\therefore \text{ P.I. } = \frac{1}{f(\theta)} x \text{ where } f(\theta) = a\theta(\theta-1) + b\theta + c$$

Let α_1 and α_2 be the roots of $f(\theta) = 0$

$$\therefore f(\theta) = a(\theta-\alpha_1)(\theta-\alpha_2)$$

$$\therefore \text{ PI } = \frac{1}{a(\theta-\alpha_1)(\theta-\alpha_2)} x.$$

$$= \left[\frac{A}{\theta-\alpha_1} + \frac{B}{\theta-\alpha_2} \right] x \text{ where A and B are constants.}$$

$$\text{To find } \frac{1}{\theta-\alpha} x.$$

$$\text{Let } \frac{1}{\theta - \alpha} x = u$$

$$(\theta - \alpha) u = x$$

$$(i.e) \quad x \cdot \frac{du}{dx} - \alpha u = x$$

$$\frac{du}{dx} - \frac{\alpha u}{x} = \frac{x}{x} \text{ which is linear in } u.$$

$$\therefore ux^{-\alpha} = \int \frac{x}{x} \cdot x^{-\alpha} dx$$

$$= \int x \cdot x^{-\alpha-1} dx$$

$$\therefore \frac{1}{\theta - \alpha} x = x^{\alpha} \int x x^{-\alpha-1} dx$$

Example :

$$\text{Solve } x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - 3y = x^2$$

$$\text{Putting } Z = \log x \text{ and } D = \frac{d}{dz}$$

$$\text{we have } x \cdot \frac{dy}{dx} = Dy \text{ and } x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

$$\therefore \text{The equation becomes } (D^2 - D)y + Dy - 3y = e^{2x}$$

$$\text{The A.E is } m^2 - 3 = 0$$

$$m = \pm \sqrt{3}$$

$$\text{C.F.} = A e^{\sqrt{3}z} + B e^{-\sqrt{3}z}$$

$$= A x^{\sqrt{3}} + B x^{-\sqrt{3}}$$

$$\text{P.I} = \frac{1}{D^2 - 3} e^{2x}$$

$$= e^{2z}. \text{ Replacing } D \text{ by } 2.$$

$$\therefore \text{The solution is } y = A x^{\sqrt{3}} + B x^{-\sqrt{3}} + x^2$$

Example :

$$\text{Solve } x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - 3y = x^2 \log x.$$

Putting $z = \log x$ and $D = \frac{d}{dz}$ we have

$$x \frac{dy}{dx} = Dy \text{ and } x^2 \frac{d^2 y}{dx^2} = D(D-1)y.$$

\therefore The equation becomes $(D^2 - 2D - 3)y = e^{2z}$

$$\therefore \text{A.E is } m^2 - 2m - 3 = 0$$

$$\therefore (m-3)(m+1) = 0$$

$$\therefore m = 3 \text{ and } -1$$

$$\text{C.F.} = Ae^{3z} + Be^{-z}$$

$$= Ax^3 + \frac{B}{x}$$

$$\text{P.I} = \frac{1}{D^2 - 2D - 3} e^{2z} z$$

$$= e^{2z} z \frac{1}{(D+2)^2 - 2(D+2) - 3}$$

replacing D by $D+2$.

$$= e^{2z} z \frac{1}{D^2 + 2D - 3}$$

$$= e^{2z} z \left(-\frac{1}{3}\right) \frac{1}{1 - \frac{2}{3}D + \frac{1}{3}D^2}$$

$$= -\frac{e^{2z} z}{3} \left(\frac{1}{1 - \frac{2}{3}D + \frac{1}{3}D^2} \right)^{-1}$$

$$= -\frac{e^{2z}}{3} \left(1 + \frac{2}{3}D \right) z$$

$$= \frac{-e^{2z}}{3} \left(z + \frac{2}{3} \right)$$

$$= \frac{-x^2}{3} \left(\log x + \frac{2}{3} \right)$$

$$\therefore \text{The solution is } y = Ax^3 + \frac{B}{x} - \frac{x^2}{3} \left(\log x + \frac{2}{3} \right)$$

Example :

$$\text{Solve } x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = ex$$

$$\text{Putting } Z = \log x \text{ and } D = \frac{d}{dz}$$

$$x \frac{dy}{dz} = Dy \text{ and } x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

$$\therefore \text{The equation becomes } (D^2 + 3D + 2) y = e^x$$

$$\text{The A.E. is } m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1 \text{ and } m = -2$$

$$\therefore \text{C.F.} = Ae^{-z} + Be^{-2z}$$

$$= \frac{A}{x} + \frac{B}{x^2}$$

$$\text{P.I} = \frac{1}{D^2 + 3D + 2} e^x$$

$$= \frac{1}{(\theta + 1)(\theta + 2)} e^x \text{ where } D = \theta = x \cdot \frac{d}{dx}$$

$$= \left[\frac{1}{\theta + 1} - \frac{1}{\theta + 2} \right] e^x$$

$$= \frac{1}{\theta + 1} e^x - \frac{1}{\theta + 2} e^x$$

$$= x^{-1} \int e^x x^{1-1} dx - x^{-2} \int e^x x^{2-1} dx$$

$$= \frac{1}{x} \int e^x dx - \frac{1}{x^2} \int x e^x dx$$

$$= \frac{1}{x} e^x - \frac{1}{x^2} (xe^x - e^x)$$

$$= \frac{e^x}{x^2}$$

The solution is $y = \frac{A}{x} + \frac{B}{x^2} + \frac{e^x}{x^2}$

Exercises : Solve the following equations :-

1) $x^2 \frac{d^2 y}{dx^2} + 8x \frac{dy}{dx} + 12y = x^4$

2) $x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = x^2$

3) $x^3 \frac{d^2 y}{dx^2} - 2x^2 \frac{dy}{dx} + 2xy = \frac{1}{4}$

4) $x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = \log x.$

7.2. Equations reducible to the linear homogeneous equations

Consider the equation of the form

$$(\ell x + m)^2 \frac{d^2 y}{dx^2} + (\ell m + m) a \frac{dy}{dx} + by = x$$

where a, b, ℓ, m are constants and x is any function of x putting $z = \ell x + m$ we get

$$Z^2 \frac{d^2 y}{dz^2} + \frac{az}{\ell} \frac{dy}{dz} + \frac{b}{\ell^2} Z = f(z)$$

which is a linear homogeneous and can be solved.

Example : Solve $(x+a)^2 \frac{d^2 y}{dx^2} - 4(x+a) \frac{dy}{dx} + 6y = x$

Putting $z = x + a$

$$\frac{dz}{dx} = 1$$

$$x = z - a.$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx} = \frac{d^2 y}{dz^2}$$

\therefore The equation becomes

$$Z^2 \frac{d^2 y}{dz^2} - 4Z \frac{dy}{dz} + 6y = z - a$$

This is a linear homogeneous equation.

Putting $u = \log z$ and $D \equiv \frac{d}{du}$ the equation

$$\text{becomes } (D^2 - D)y - 4Dy + 6y = e^u - a$$

$$(\text{i.e.}) (D^2 - 5D + 6)y = e^u - a$$

$$\text{The A.E is } m^2 - 5m + 6 = 0$$

$$\therefore m = 2 \quad \text{and} \quad m = 3$$

$$\begin{aligned}\therefore \text{The C.F.} &= Ae^{2u} + Be^{3u} \\ &= Az^2 + Bz^3 \\ &= A(x+a)^2 + B(x+a)^3\end{aligned}$$

$$P.I = \frac{e^u - a}{D^2 - 5D + 6}$$

$$= \frac{e^u}{D^2 - 5D + 6} - \frac{a}{D^2 - 5D + 6}$$

$$= \frac{e^u}{2} - \frac{a}{6}$$

$$= \frac{z}{2} - \frac{a}{6}$$

$$= \frac{x+a}{2} - \frac{a}{6}$$

$$\begin{aligned}\therefore \text{The solution is } y &= A(x+a)^2 + B(x+a)^3 \\ &+ \frac{(x+a)}{2} - \frac{a}{6}\end{aligned}$$

Exercise :

Solve :

$$1) (5+2x)^2 \frac{d^2y}{dx^2} - 6(5+2x) \frac{dy}{dx} + 8y = 6x.$$

$$2) (x+1)^2 \frac{d^2y}{dx^2} - 3(x+1) \frac{dy}{dx} + 4y = x^2 + x + 1$$

$$3) (1+2x)^2 \frac{d^2y}{dx^2} - (1+2x) \frac{dy}{dx} + y = 8(1+2x)^2$$

7.3. Simultaneous Differential equation

We now consider systems of equations. There is only one independent variable usually denoted by the symbol t and a number of dependent variables equal to the number of equations. We shall further suppose that the equations are linear.

Let D stand for $\frac{d}{dt}$. Taking the simplest case of two independent variables x and y , the equations can be written in the form.

$$f_1(D)x + \phi_1(D)y = T_1 \dots\dots\dots(1)$$

$$f_2(D)x + \phi_2(D)y = T_2 \dots\dots\dots(2)$$

Where we shall suppose that f_1, f_2, ϕ_1, ϕ_2 are rational integral functions of D with constant coefficient T_1, T_2 explicit functions of t . To eliminate y , as in solving simultaneous algebraic equations we operate on (1) by $\phi_2(D)$ and (2) by $\phi_1(D)$ and subtract. Then we have

$$[f_1(D)\phi_2(D) - f_2(D)\phi_1(D)]x = \phi_2(D)T_1 - \phi_1(D)T_2$$

This can be solved by the method of the another method. Substituting for x in (1) or (2) y can be found.

It must be noted that the number of arbitrary constants in the complete solution is the exponent of the highest order in the operator D of $f_1(D)\phi_2(D) - f_2(D)\phi_1(D)$.

The above method can be extended to equations of more than two dependent variables. The following examples illustrate the process clearly.

Examples :

$$\text{Solve } \frac{dx}{dt} + 2x - 3y = t \dots\dots\dots(1)$$

$$\frac{dy}{dt} - 3x + 2y = e^{2t} \dots\dots\dots(2)$$

The equation can be written as $(D+2)x - 3y = t$ and $(D+2)y - 3x = e^{2t}$.

To eliminate y , operate on (1) by $D+2$ and (2) by 3 and add

we get

$$[(D+2)^2-9] x = (D+2)t + 3e^{2t}$$

$$(i.e) (D^2+4D-5)x = 1+2t+3e^{2t}$$

The auxiliary equation is $m^2+4m-5=0$ $m=1$ (or) -5

$$C.F. \text{ is } c_1 e^t + c_2 e^{-5t}$$

$$P.I = \frac{1}{D^2+4D-5} (1+2t) + \frac{3e^{2t}}{7}$$

$$= \frac{1}{5} \left[1 + \frac{4D}{5} \right] (1+2t) + \frac{3e^{2t}}{7}$$

$$= \frac{-13}{25} - \frac{2t}{5} + \frac{3e^{2t}}{7}$$

$$\therefore x = c_1 e^t + c_2 e^{-5t} - \frac{13}{25} - \frac{2t}{5} + \frac{3e^{2t}}{7}$$

Substituting this value of x in (1).

$$y = c_1 e^t - c_2 e^{-5t} - \frac{12}{25} - \frac{3t}{5} + \frac{4e^{2t}}{7}$$

Note :

If we had substituted the value of x in (2) y would be given by a first order linear differential equation and the integration of this will give rise to a third constants C_3 besides C_1 and C_2 . They cannot be arbitrary as the number of arbitrary constants is the only 2 the exponent of the highest power of D in D^2+4D-5 . In fact the relation connecting C_1, C_2, C_3 in this case to be found by substituting the values of x and y in (1).

Example : Solve $\frac{dx}{dt} = ax + by + c$

$$\frac{dy}{dt} = a^1 x + b^1 y + c^1$$

This can be solved by the ordinary method. But the following elegant method can be adopted in equation of this form.

Multiplying the second equation by m and adding to the first (m to be suitably determined).

$$\frac{d}{dt} (x+my) = x(a+ma^1) + y(b+mb^1) + c+mc^1 \dots\dots\dots(1)$$

Close m in such a way that

$$\frac{a+ma^1}{1} = \frac{b+mb^1}{m} = k \text{ (say)}$$

$\therefore m$ is a root of the quadratic $m^2a^1 + m(a-b^1) - b = 0$

Let m_1 and m_2 be the roots of this quadratic.

The equation (1) becomes linear in $(x+my)$ viz.

$$\frac{d}{dt} (x+my) - k (x+my) = c+mc^1$$

$$\therefore (x+my) e^{-kt} = \frac{(c+mc^1) e^{-kt}}{-k} + A$$

$$\therefore x+my = \frac{-c+mc^1}{k} + Ae^{kt} \dots\dots\dots(2)$$

where $m=m_1$ and m_2 and k has the respective values $a+m_1a^1$ and $a+m_2a^1$ (2) represents the two equations which constitute the solution of the given pair of equations.

Example : Solve $4 \frac{dx}{dt} + 9 \frac{dy}{dt} + 2x + 31y = e^t$

$$3 \frac{dx}{dt} + 7 \frac{dy}{dt} + x + 24y = 3$$

The equations can be written as

$$2 (2D+1)x + (9D+31) y = e^t$$

$$(3D+1)x + (7D+24) y = 3$$

$$\text{Eliminating } (D^2+8D+17) x = 31 (e^t-3)$$

The auxiliary equation is $m^2 + 8m + 17 = 0$

$$\therefore m = -4 \pm i$$

$$\text{C.F.} = e^{-4t} (c_1 \sin t + c_2 \cos t)$$

$$\text{P.I} = \frac{1}{D^2 + 8D + 17} 31 (e^t - 3) = \frac{31e^t}{26} - \frac{93}{17}$$

$$\therefore x = e^{-4t} (C_1 \sin t + C_2 \cos t) + \frac{31}{26} e^t - \frac{93}{17}$$

Similarly eliminating x , we have

$$(D^2 + 8D + 17) y = 5 - 4e^t$$

$$y = e^{-4t} (C_3 \cos t + C_4 \sin t) + \frac{6}{17} - \frac{2e^t}{13}$$

The relation between the constants are got by substituting these values in the two equation substituting in (1).

$$\sin t (-14C_1 - 4C_2 - 5C_4 + 9C_3) + \cos t (-14C_3 + 4C_1 + 9C_4 - 5C_2) = 0$$

$$\text{Hence } C_3 = -(C_1 + C_2) \text{ and } C_4 = C_2 - C_1$$

$$y = \left[(C_2 - C_1) \sin t - (C_2 + C_1) \cos t \right] e^{-4t} - \frac{2e^t}{13} + \frac{6}{17}$$

Exercises :

1) $3(1-D)x + 4y = 3t + 1; \quad 3(D+1)y + 2x = e^t$

2) $(5+D)x + y = e^t; \quad (D+3)y - x = e^{2t}$

3) $(D+8)x + (2D+1)y = 0; \quad (6D-2)x + (3D-11)y = 0$

UNIT - 8

LINEAR EQUATION OF THE SECOND ORDER

8.1 : Complete solution given a Known integral:

If an integral included in the complementary function of the given equation be known the complete solution can be found in terms of this known integral.

$$\text{Let } \frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R \quad (1)$$

be the given equation where P, Q, R are functions of x.

Let $y = y_1$ be a known integral in the c.f of (1)

$$(ie) \quad \frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = 0$$

$$\frac{d^2y_1}{dx^2} + P \frac{dy_1}{dx} + Qy_1 = 0 \quad (2)$$

Putting $y = y_1 v$ in (1) where v is a function of x.

$$\text{we get } y_1 \left[\frac{d^2y}{dx^2} + \frac{dy}{dx} \left(2 \frac{dy_1}{dx} + py_1 \right) \right] = R \text{ in limit } (2)$$

This is linear in $\frac{dy}{dx}$; hence

$$\frac{dy}{dx} = \frac{c_1}{y_1^2} e^{-\int p dx} + \frac{e^{-\int p dx}}{y_1^2} \int Ry_1 e^{\int p dx} dx$$

Integrating

$$v_1 = c_2 + c_1 \int \frac{e^{-\int p dx}}{y_1^2} dx + \int \left\{ \frac{e^{-\int p dx}}{y_1^2} \int Ry_1 e^{\int p dx} dx \right\} dx$$

The solution of (1) is $y = vy_1$ where v has the above value. It must be noted that this solution includes the given solution and that there are two arbitrary constants.

Remark : Some cases where in simple function of x, like x and e^x are integrals of

the equations $P_2 = \left(\frac{d^2y}{dx^2} \right) + P_1 \frac{dy}{dx} + P_0 y = 0$ should be noted.

Thus for the above equation.

$$y = x ; y = e^x : y = e^{-x} \text{ and } y = x^2 \text{ are solution if}$$

$$P_1 + P_0 x = 0$$

$$P_2 + P_0 = 0$$

$$P_2 - P_1 + P_0 = 0$$

$$2P_2 + 2P_1 x + P_0 x^2 = 0 \text{ respectively.}$$

Examples:-

$$1) \text{ Solve } x \frac{d^2 y}{dx^2} - (2x-1) \frac{dy}{dx} + (x-1)y = e^x$$

As the sum of the coefficients of the first member zero. e^x is a solution of

$$x \frac{d^2 y}{dx^2} - (2x-1) \frac{dy}{dx} + (x-1)y = 0$$

$$\text{Put } y = ve^x \text{ (i) reduces to } x \frac{d^2 v}{dx^2} + \frac{dv}{dx} = 1$$

$$\text{solving } \frac{dv}{dx} x = x + c, \text{ (ie) } \frac{dv}{dx} = 1 + \frac{c_1}{x}$$

$$\text{Integrating } v = x + c_1 \log x + c_2$$

$$\text{Hence } y = e^x (x + c_1 \log x + c_2)$$

$$(2) \text{ Solve } x^2 \frac{d^2 y}{dx^2} - (x^2 + 2x) \frac{dy}{dx} + (x+2)y = x^3 e^x$$

$y = x$ is a solution of this equation without the second member.

$$\text{Putting } y = vx \text{ it reduces to } v_2 - v_1 = e^x$$

$$\text{Hence } vie^{-x} = x = c_1 \text{ (or) } v_1 = (x+c_1) e^x$$

$$\text{Integrating } v = (c_1 - 1) e^x = xe^x + c_2$$

$$\therefore y = c_2 x + (c_1 - 1) xe^x + x^2 e^x$$

8.2 : Reduction to the normal form:

$$\text{Consider } \frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R \quad (1)$$

Putting $y = y_1 v$ this becomes

$$y_1 \frac{d^2 v}{dx^2} + \frac{dv}{dx} \left[2 \frac{dy_1}{dx} + Py_1 \right] + v \left[\frac{d^2 y_1}{dx^2} + \frac{pdy_1}{dx} + Qy_1 \right] = R$$

If y_1 be chosen to satisfy $\frac{dy_1}{dx} + Py_1 = 0$

(ie) $y_1 = e^{-\int P dx}$ then the above equation becomes $\frac{d^2y}{dx^2} + Iv = Re^{-\int P dx}$ (2)

where $I = Q - \frac{1}{2} \frac{dP}{dx} - \frac{P^2}{4}$

(2) is immediately integrable if I be either a constant (or) $\frac{a}{x^2}$ where a is

constant. This method is either called reducing (1) the normal form (or) removing the first derivative.

Example :

1) Solve $y_2 - 4xy_1 + (4x^2 - 3)y = e^{x^2}$

Here $P = -4x$; $Q = 4x^2 - 3$ and $R = e^{x^2}$

$$y_1 = e^{-\int P dx} = e^{x^2}$$

Putting $y = vy_1$ $I = Q - \frac{1}{2} \frac{dP}{dx} - \frac{P^2}{4} = -1$ and the equation reduces to $\frac{d^2v}{dx^2} - v = 1$

Hence $v = Ae^x + Be^{-x} - 1$

$$\therefore y = ex^2 (Ae^x + Be^{-x} - 1)$$

2) $4x^2 \frac{d^2y}{dx^2} + 4x^5 \frac{dy}{dx} + (x^8 + 6x^4 + 4)y = 0$

Here $P = x^3$; $Q = \frac{1}{4} (x^6 + 6x^2 + \frac{4}{x^2})$

$$y_1 = e^{-\int P dx} = e^{-\frac{x^4}{4}}$$

$$I = Q - \frac{1}{2} \frac{dP}{dx} - \frac{P^2}{4} = \frac{1}{x^2}$$

Hence the equation in v; where $y = vy_1$ is

$$\frac{d^2v}{dx^2} + \frac{1}{x^2} v = 0$$

This is a homogeneous linear equation whose solution is $\sqrt{x} A \cos\left(\frac{\sqrt{3}}{2} \log x + B\right)$

$$\therefore y = \frac{-x^4}{e^8 \sqrt{x}} A \cos\left(\frac{\sqrt{3}}{2} \log x + B\right)$$

8.3 Variation of parameters:-

consider $\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$ ----- (1)

Let y_1 be a solution of $\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = 0$ ----- (2)

so that $\frac{d^2y_1}{dx^2} + P \frac{dy_1}{dx} + Qy_1 = 0$ ----- (3)

Eliminating (1) between (2) and (3) we get

$$y_1 \frac{d^2y}{dx^2} - y \frac{d^2y_1}{dx^2} + P \left(y_1 \frac{dy}{dx} - y \frac{dy_1}{dx} \right) = 0$$

The integral is $y_1 \frac{dy}{dx} - y \frac{dy_1}{dx} = A e^{-\int p dx}$

Integrating $y = By_1 + Ay_1 \int \frac{e^{-\int p dx}}{y_1^2} dx$

If y_2 denotes $y_1 \int \frac{e^{-\int p dx}}{y_1^2} dx$ the above relation .

$y = By_1 + Ay_2$ Proves that y_2 is a solution of (1) without the second member. we deduce from the above that the two particular solution y_1 and y_2 are connected by the relation.

$$y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} = c e^{-\int p dx} \text{ ----- (I)}$$

where c_1 is a determinate (not arbitrary) constant depending on y_1 and y_2 .

Let us substitute $y = Ay_2 + By_1$ in (I) on the hypothesis that A and B are no longer constants but functions of x to be so chosen as to satisfy (I). Though the form of y is the same for the equation (I) and

$$\frac{d^2y}{dx^2} + p \frac{dy}{dx} + Qy = 0 \text{ the constants A and B occurring in the solution}$$

of the latter equation change into function of x in the solution of the former. Hence this process is called "variation of parameter" . Now $y = Ay_2 + By_2$.

$$\begin{aligned}\frac{dy}{dx} &= A \frac{dy_2}{dx} + y_2 \frac{dA}{dx} + B \frac{dy_1}{dx} + y_1 \frac{dB}{dx} \\ &= B \frac{dy_1}{dx} + A \frac{dy_2}{dx}\end{aligned}$$

provided we impose the relation.

$$y_1 \frac{dB}{dx} + y_2 \frac{dA}{dx} = 0 \quad (4)$$

on the two quantities A and B which we are at liberty to do.

Differentiating

$$\frac{d^2y}{dx^2} = B \frac{d^2y_1}{dx^2} + A \frac{d^2y_2}{dx^2} + \frac{dB}{dx} \frac{dy_1}{dx} + \frac{dA}{dx} \frac{dy_2}{dx}$$

substituting these values in (1) and noting that y_1 and y_2 are particular solutions of (1) without the second member.

$$\text{We get } \frac{dB}{dx} \frac{dy_1}{dx} + \frac{dA}{dx} \frac{dy_2}{dx} = R$$

From (4) and this equation

$$\begin{aligned}\frac{dA}{dx} &= \frac{dB}{dx} = \frac{R}{y_1 - y_2} \\ &= \frac{R}{y_1} \frac{dy_2}{dx} - \frac{R}{y_2} \frac{dy_1}{dx}\end{aligned}$$

$$= \frac{R}{c} e^{\int p dx} \quad \text{by (1)}$$

$$\therefore A = a + \frac{1}{c} \int R y_1 e^{\int p dx} dx$$

$$B = b + \frac{1}{c} \int R y_2 e^{\int p dx} dx$$

where a and b are arbitrary constants and c is a defined constant depending on y_1 and y_2 .

$$\therefore y = A y_2 + B y_1 \text{ is the required solution of (1).}$$

It is usual to write the above results in the following form setting $P = \phi(x)$:

$R = \psi(x)$ $y_1 = f_1(x)$ $y_2 = f_2(x)$ the solution of

$$\frac{d^2y}{dx^2} + \phi(x) \frac{dy}{dx} + Qy = \psi(x) \text{ is}$$

$$y = a f_2(x) + b f_1(x) + \frac{1}{c} \int_e^x \psi(x) \phi(x) dx$$

$$\{f_2(x) f_1(x) - f_1(x) f_2(x)\} dx$$

where $f_1(x)$ and $f_2(x)$ are solutions of

$$\frac{d^2y}{dx^2} + \phi(x) \frac{dy}{dx} + Qy = 0 \text{ and are connected by}$$

$$f_1 \frac{df_2}{dx} - f_2 \frac{df_1}{dx} = - \int_e^x \phi(x) dx$$

without loss of generality we can take to be unity by suitably adjusting $f_2(x)$ for instance.

Examples:

$$1) \text{ Solve } \frac{d^2y}{dx^2} + n^2y = \sec(nx) \quad (1)$$

The solution of (1) without the second member is $y = A \cos(nx) + B \sin(nx)$

Assume this to be a solution of (1). On the supposition that A and B are no

longer constants $\frac{dy}{dx} = -A_n \sin nx + B_n \cos nx$ provided we impose upon A

and B the relation $\frac{dA}{dx} \cos nx + \frac{dB}{dx} \sin nx = 0$.

$$\frac{d^2y}{dx^2} = -An^2 \cos nx - Bn^2 \sin nx - n \sin nx \frac{dA}{dx} + n \cos nx \frac{dB}{dx}$$

Putting these values in (1)

$$-n \sin(nx) \frac{dA}{dx} + n \cos nx \frac{dB}{dx} = \sec(nx)$$

$$\text{Now } \frac{dA}{dx} = \frac{dB}{dx} = \frac{\sec nx}{n}$$

$$\frac{dA}{-\sin(nx)} = \frac{dB}{\cos nx}$$

$$\text{Hence } A = a + \frac{1}{n^2} \log (\cos nx)$$

$$B = b + \frac{x}{n}$$

$$\therefore y = a \cos nx + b \sin nx + \frac{\cos nx}{n^2} \log (\cos nx) + \frac{x}{n} \sin (nx)$$

where a and b are arbitrary constants.

$$2) (1-x) y_3 + (x^2-1)y_2 - x^2y_1 + xy = 0 \text{ ----- (1)}$$

$y = x$ and $y = e^x$ are solutions by inspection

put $y = Ax + Be^x$ where A and B are parameters

$$y_1 = A + Be^x + x \frac{dA}{dx} + x e^x \frac{dB}{dx}$$

Choose A and B to satisfy the condition.

$$x \frac{dA}{dx} + e^x \frac{dB}{dx} = 0 \text{ ----- (i)}$$

so that $y_1 = A + Be^x$

$$y_2 = \frac{dA}{dx} + Be^x + e^x \frac{dB}{dx}$$

$$y_3 = \frac{d^2A}{dx^2} + Be^x + 2e^x \frac{dB}{dx} + e^x \frac{d^2B}{dx^2}$$

substituting in the given equation (1)

$$(1-x) \frac{d^2A}{dx^2} + e^x (1-x) \frac{d^2y}{dx^2} + (x^2-1) \frac{dA}{dx} + e^x (x-1)^2 \frac{dB}{dx} = 0$$

$$\text{(ie)} \frac{d^2A}{dx^2} + e^x \frac{d^2B}{dx^2} + e^x(x-1) \frac{dB}{dx} - (1+x) \frac{dA}{dx} = 0$$

$$(ie) \frac{d^2A}{dx^2} + e^x \frac{d^2B}{dx^2} + (x^2 - 2x - 1) \frac{dA}{dx} = 0 \text{ by (i) } \text{-----} (2)$$

$$\text{Differentiating (i) } x \cdot \frac{d^2A}{dx^2} + \frac{dA}{dx} + e^x \left(\frac{d^2B}{dx^2} + \frac{dB}{dx} \right) = 0 \text{ ----- (ii)}$$

Eliminating (B) by (i) and (ii), (iii) reduces to

$$\frac{d^2A}{dx^2} = \frac{x^2 - x + 2}{x - 1} \frac{dA}{dx}$$

Integrating

$$\log \frac{dA}{dx} = \frac{x^2}{2} - 2 \log (x - 1) + \log c_1$$

$$\therefore \frac{dA}{dx} = \frac{c_1 \frac{x^2}{e^2} + x}{(x - 1)^2}$$

$$\text{From (i) } \frac{dB}{dx} = - \frac{C_1 x \frac{x^2}{e^2} - x}{(x - 1)^2}$$

$$\therefore A = c_1 \int \frac{e}{(x - 1)^2} dx + c_2$$

$$\therefore B = - c_1 \int \frac{xe}{(x - 1)^2} dx + c_3$$

Hence $y = Ax + Be^x$ where A and B have the above values (c_1, c_2, c_3 are arbitrary constants)

8.4 Total differential equations

In a total differential equation. We have the differential coefficients of several dependent valuables with reference to a single independent variable such an equation in three variables is represented

$$Pdx + Qdy + Rdz = 0 \text{ ----- (1)}$$

Criterion of integrability:-

$$\text{Let (i) have an integral } v(x, y, z) = a \text{ ----- (2)}$$

where a is an arbitrary constant. Then differentiating (2) totally

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

This is identical with (1)

$$\text{Hence } \frac{\partial u}{\partial x} = \mu P ; \frac{\partial u}{\partial y} = \mu Q ; \frac{\partial u}{\partial z} = \mu R$$

$$\frac{\partial}{\partial y} (\mu P) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} (\mu Q)$$

$$\therefore \mu \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = Q \left(\frac{\partial \mu}{\partial x} \right) - P \left(\frac{\partial \mu}{\partial y} \right) \text{----- (i)}$$

$$\text{Similarly } \therefore \mu \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) = R \frac{\partial \mu}{\partial y} - Q \frac{\partial \mu}{\partial z} \text{----- (ii)}$$

$$\text{and } \mu \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) = P \frac{\partial \mu}{\partial z} - R \frac{\partial \mu}{\partial x} \text{----- (iii)}$$

Multiplying (1), (2), (3) by R, P and Q Respectively and adding we obtain.

$$P \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = 0 \text{----- (I)}$$

The above condition is spoken of as the condition of integrability of equation (I)

We have just shown that the above condition (I) is necessary for the existence of the integral of equation (I)

We shall show that the condition is sufficient by proving that an integral of (i) can be found when (I) holds good.

Result :

If relation (I) exists among P, Q, R a similar relation holds good between the coefficients of

$$\mu P dx + \mu Q dy + \mu R dz = 0 \text{----- (II)}$$

Where μ is any function of x, y, z

If $P dx + Q dy$ (where z is a parameter) is not an exact differential. We can always find an integration factor μ such that $\mu P dx + \mu Q dy$ is exact. Hence without loss of generality. $P dx + Q dy$ can be regarded as exact differential and (III) as the equation to be considered.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Let $V = \int Pdx + Qdy$

(ie) $\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = Pdx + Qdy$

Hence $\frac{\partial v}{\partial x} = P$ and $\frac{\partial v}{\partial y} = Q$

$\frac{\partial v}{\partial x} = \frac{\partial^2 v}{\partial z \partial x}$ and $\frac{\partial Q}{\partial z} = \frac{\partial^2 v}{\partial z \partial y}$

(I) now reduces to

$$\frac{\partial v}{\partial x} \left(\frac{\partial^2 v}{\partial z \partial y} - \frac{\partial R}{\partial y} \right) + \frac{\partial v}{\partial y} \left(\frac{\partial R}{\partial x} - \frac{\partial^2 v}{\partial z \partial x} \right) = 0$$

$$\left[\begin{array}{cc} \frac{\partial v}{\partial x} & \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial z} - R \right) \\ \frac{\partial v}{\partial y} & \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial z} - R \right) \end{array} \right] = 0$$

(ie) $\frac{\partial}{\partial (x,y)} \left(v_1 \frac{\partial v}{\partial z} - R \right) = 0$

\therefore A relation independent of x and y exists between V and $\frac{\partial v}{\partial z} - R$ and

hence $\frac{\partial v}{\partial z} - R$ can be expressed as a function of z and v .

Let $\frac{\partial v}{\partial z} - R = \phi(z, v)$

Since $Pdx + Qdy + Rdz$

$$= \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz + \left(R - \frac{\partial v}{\partial z} \right) dz$$

$$= dv - \phi(z, v) dz$$

This being an equation in two variables z and v leads on integration to $f(v, z) = 0$. Hence (I) is the necessary and sufficient criterion for integrability of (I).

Rule for integrating : $Pdx + Qdy + Rdz = 0$

When the condition of integrability is satisfied consider one of the variables say z , as constant $dz=0$ and integrate the equation $Pdx + Qdy = 0$. Put the arbitrary constant of integration that occurs in this integral as an arbitrary function of z . This is justified as the arbitrary constant in this integral is a constant usually with respect to x and y differentiating this integral with respect to x, y and z and comparing the result with (1) we can determine the arbitrary function of z . The following examples will illustrate the process.

Examples:

1) Solve $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$ ----- (1)

Here the condition of integrability is satisfied neglecting $(y^2 - xy)dz$ we get

$$(y^2 + yz)dx + (xz + z^2)dy = 0$$

$$\frac{dx}{x+z} + \frac{zdy}{y(y+z)} = 0$$

Integrating on the assumption that z is a constant.

$$\log(x+z) + \log \frac{y}{y+z} = \log c.$$

$$(ie) \frac{(x+z)y}{y+z} = c = f(z) \text{ ----- (2)}$$

Where the arbitrary constant c of integration is put as $f(z)$, an arbitrary function of z . Differentiating (2) totally with respect to x, y, z .

$$\frac{(y+z)[(dx+dz)y + (x+z)dy] - y(x+z)(dy+dz)}{(y+z)^2} = f'(z) dz$$

(ie) $(y^2 + yz)dx + (xz + z^2)dy + dz(y^2 - xy - f'(z)(y+z)^2)$ comparing this with (1) we have $f'(z)(y+z)^2 dz = 0$ As $(y+z)^2 dz \neq 0$; $f'(z) = 0$ $f(z)$ is constant c . Hence the integral of (1) is $y(x+z) = c(y+z)$.

Example :

Show that the equation.

$$(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0 \text{ ----- (1) satisfies the condition of integrability and solve it.}$$

Pf: The condition of integrability is easily verified. We neglect the dy term as the coefficient of dz is a simpler than that of dy and have retained. (ie) treat y as a constant for the time being.

Then we have

$$(x^2y - y^3 - y^2z) dx + xy(y+x) dz = 0$$

$$\frac{dz}{dx} - \frac{yz}{(x+y)x} = \frac{y-z}{x}$$

This is linear in z and hence I.F is

$$e^{-\int \frac{y dx}{x(x+y)}} = \frac{x+y}{x}$$

$$ze^{-\int \frac{y dx}{x(x+y)}} = \int \frac{(y-x)}{x} \cdot \frac{(x+y)}{x} \cdot dx + c$$

$$\text{ie } z \cdot \frac{x+y}{x} = -x - \frac{y^2}{x} + f(y)$$

Where c is put as an arbitrary function of y .

$$\therefore zx + yz + x^2 + y^2 = xf(y) \quad \text{----- (2)}$$

Differentiating (2) totally we get

$$dx [z + 2x - f(y)] + dy [z + 2y - xf'(y)] + dz [x + y] = 0$$

multiplying this by xy and subtracting (1) from the multiplied result.

$$dx [xyz + x^2y - xyf(y) + y^3 + y^2z] + dy [xyz + xy^2 - x^2y + f'(y) + x^2z + x^3] = 0$$

Using (2) this reduces to $dy [x^2(fy) - x^2yf'(y)] = 0$

$$\text{As } dy \neq 0 - y f'(y) + f(y) = 0 \quad \therefore \frac{df}{f} = \frac{dy}{y}$$

Hence $f = cy$ where c is an arbitrary constant.

The solution of (1) is this $xz + yz + x^2 + y^2 + cxy$.

Exercises:

1) verify the condition of integrability in the following equations and solve them.

1) $(y+z)dx + (z+x)dy + (x+y)dz = 0$

2) $(y+z)dx + dy + dz = 0$

3) $(2x^2 + 2xy + 2xz^2 + 1) dx + dy + 2zdz = 0$.

UNIT - 9

9.1. Partial differential Equations :

A Partial differential Equation is one which involves at least two independent variables. We shall consider only equations with three variables x, y and z where x and y are independent.

Let the symbols P and Q denote respectively the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Then a partial differential equation can be that in the form $F(x, y, z, p, q) = 0$.

Formation of partial Differential Equations :

There are two ways of obtaining a partial differential equation (i) by eliminating the arbitrary constants (ii) by eliminating the arbitrary functions.

By eliminating the arbitrary Constants :

Suppose $\phi(x, y, z, a, b) = 0 \rightarrow (1)$ be such a relation.

Differentiating partially W.r. to x and y .

$$\frac{\partial \phi}{\partial x} + P \frac{\partial \phi}{\partial z} = 0 \text{ and } \frac{\partial \phi}{\partial y} + Q \frac{\partial \phi}{\partial z} = 0$$

$$\frac{\partial \phi}{\partial x} + P \frac{\partial \phi}{\partial z} = 0 \text{ and } \frac{\partial \phi}{\partial y} + Q \frac{\partial \phi}{\partial z} = 0 \rightarrow (2)$$

using (1) and (2) we can eliminate the constants a and b and get a differential equation of the form $F(x, y, z, p, q) = 0$.

Note :

When the number of constants eliminated is the same as the number of independent variables the equation (2) will be a partial differential equation of the first order.

When the number of constants eliminated is the greater than the number of independent variables, then the resulting equation will be of higher order.

Example :

1. From $Z = ax + by + ab$ obtain a partial differential equation.

$$Z = ax + by + ab$$

$$\therefore \frac{\partial Z}{\partial x} = a \text{ and } \frac{\partial Z}{\partial y} = b$$

$$\therefore P = a \text{ and } q = b$$

Substituting for a and b

$$Z = Px + qy + pq.$$

2) Eliminate h and k from

$$(x-h)^2 + (y-k)^2 + Z^2 = C^2.$$

Differentiating partially w.r. to x.

$$2(x-h) + 2z \frac{\partial z}{\partial x} = 0.$$

$$(i.e.) (x-h) + ZP = 0.$$

$$x-h = -ZP$$

$$\text{Similarly } y-k = -Zq.$$

Substituting in the given equation.

$$Z^2P^2 + Z^2q^2 + Z^2 = c^2.$$

$$(i.e.) Z^2(P^2 + q^2 + 1) = c^2.$$

Exercises :

1. For a partial differential equation by eliminating h and K from $Z = (x^2+h)(y^2+k)$
2. Eliminate a and b from $z = ax + a^2y^2 + b$

By eliminating an arbitrary function :

Let U and V be functions of Z. Then we can find a relation between u and v in the form $\phi(u,v) = 0$ where ϕ is arbitrary. Differentiating partially w.r. to x.

$$\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} P \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} P \right) = 0 \rightarrow (1)$$

Differentiating partially w.r. to y.

$$\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \right) = 0 \rightarrow (2)$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (1) and (2) we have

$$\frac{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} P}{\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q} = \frac{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} P}{\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q}$$

Cross multiplying and collecting the terms of p and q.

$$\left[\frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} \right] P + \left[\frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z} \right] P$$

$$= \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$$

(i.e.) $P_p + Q_q = R$ where

$$P = U_y V_z - U_z V_y \quad Q = U_z V_x - U_x V_z$$

$$R = U_x V_y - U_y V_x$$

Note :

The equation $P_p + Q_q = R$ is called Lagrange's equation.

Example :

From a partial differential equation by eliminating the function ϕ from $\phi(x+y+z, x^2+y^2-z^2) = 0$

Let $u = x+y+z$ and $V = x^2+y^2-z^2$.

The required differential equation is $P_p + Q_q = R$ where,

$$P = U_y V_z - U_z V_y = 1(-2z) - 1(2y)$$

$$= -2(z+y)$$

$$Q = U_z V_x - U_x V_z = 1(2x) - 1(-2z) = 2(x+z)$$

$$R = U_x V_y - U_y V_x = 1(2y) - 1(2x) = 2(y-x)$$

The partial differential equation is

$$-2(z+y)P + 2(x+z)q = 2(y-x)$$

(i.e.) $(z+y)P - (x+z)q = x-y$

Exercises :

1. Form a partial differential equation by eliminating the function f from,

$$Z = y^2 + 2F\left[\frac{1}{x} + \text{Log } Y\right]$$

2. Eliminate f and ϕ from the relation $Z = f(x+iy) + \phi(x-iy)$

3. Eliminate ϕ from the relation $\phi(x+y+z, xyz) = 0$.

9.2. Classification of Integrals :

Complete integral, particular integral, general integral and singular integral :

We saw that if there is a relation of the form $\phi(x, y, z, a, b) = 0$ (1) where a, b are arbitrary constants, by eliminating a and b we get a differential equation of the form, $F(x, y, z, p, q) = 0 \rightarrow (2)$

Which is of the first order when the number of such arbitrary constants is greater than the number of independent variables (2) is of the higher order. The relation (1) which contains as many arbitrary constants as the number of independent variables is called the complete integral of the equation (2).

Geometrically (1) represents a doubly infinite system of surfaces. For it is the set of all points (x, y, z) which satisfy the equation (1). By giving particular values for the constants a and b in the complete integral we get a particular integral. Consider the complete integral $\phi(x, y, z, a, b) = 0$ of the differential equation. $F(x, y, z, p, q) = 0$. Suppose one of the constants a and b is a function of the other, say $b = f(a)$. Then the integral becomes,

$$\phi[x, y, z, a, f(a)] = 0 \rightarrow (3).$$

For different values of a and b , the equation (1) represents a doubly infinite system of surfaces. Since $b = f(a)$ (3) represents a subset of the set of all surfaces represented by (1)

$$\text{Now consider } \frac{\partial \phi}{\partial a} = 0 \rightarrow (4)$$

(3) and (4) together represent the curve of intersection of two consecutive surfaces of the system (3). This curve is called the characteristic of the envelope of the family of surfaces (3). Hence the general integral may be defined as the locus of the characteristics.

Consider the relations :

$$\phi(x, y, z, a, b) = 0 \quad \frac{\partial \phi}{\partial a} = 0; \quad \frac{\partial \phi}{\partial b} = 0$$

Eliminate a and b from the above equations. Geometrically the eliminate will represent the envelope of the family of the surfaces represented by (1). The eliminant is called the singular integral (or) singular solution of $F(x, y, z, p, q) = 0$.

Method of solving a linear partial differential equation of the first order given in Lagrange's form $P_p + Q_q = R$.

We know that the equation $P_p + Q_q = R$ is obtained by eliminating the arbitrary function ϕ from $\phi(u, v) = 0$ where.

$$P = \frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z} - \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial z}$$

and $R = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$

Hence the general integral of $P_p + Q_q = R$ is $\phi(u, v) = 0$.

Where U and V are to be determined from the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Note :

The solution $\phi(u, v) = 0$ can also be written as $U = \psi(v)$

Example :

1. Solve $xzp + yzq = xy$

The Auxiliary equations are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

Taking the first two $\frac{dx}{x} = \frac{dy}{y}$

$$\therefore \log x = \log y + \log c_1$$

$$x = c_1 y$$

$$\frac{x}{y} = c_1 \rightarrow (1)$$

Taking the last two, $\frac{dy}{z} = \frac{dz}{x}$

(i.e.) $\frac{dy}{z} = \frac{dz}{C_1 y} = \text{from (1)}$

$$C_1 y dy = z dz$$

$$C_1 \frac{y^2}{2} = \frac{z^2}{2} + C^2$$

$$C_1 y^2 - z^2 = c$$

$$\frac{x}{y} = y^2 - z^2 = C.$$

$$(i.e.) xy - z^2 = c.$$

$$\therefore \text{The required solution is } \phi \left(xy - z^2, \frac{x}{2y} \right) = 0$$

$$(Or) xy - z^2 = \psi \left(\frac{x}{y} \right)$$

2) Solve $x(y-z)P + y(z-x)Q = z(x-y)$

The Auxiliary equations are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$\Sigma x(y-z) = 0 \quad \therefore 1, 1, 1 \text{ are multipliers.}$$

$$\therefore 1.d + 1.dy + 1.dz = 0$$

$$dx + dy + dz = 0$$

$$(i.e.) x + y + z = c_1 \rightarrow (1)$$

$$\text{Again } \therefore \Sigma \frac{1}{y-z} = 0 \quad \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \text{ are multipliers}$$

$$\therefore \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0$$

$$(i.e.) \log x + \log y + \log z = \log C_2.$$

$$(i.e.) xyz = C_2.$$

$$\therefore \text{The required solution is } \phi(x+y+z, xyz) = 0.$$

$$(or) x+y+z = \psi(xyz)$$

3) Solve $(y^2+z^2-x^2)P - 2xyQ + 2xz = 0$

$$(Y^2+z^2-x^2)P - 2xyQ = -2xz.$$

$$(i.e.) (x^2-y^2-z^2)P + 2xyQ = 2xz.$$

\therefore The Auxiliary equations are

$$\frac{dx}{(x^2 - y^2 - z^2)} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

each is equal to $\frac{xdx + ydy + zdz}{x(x^2 - y^2 - z^2) + y(2xy) + z(2xz)}$

$$= \frac{xdx + ydy + zdz}{x^3 + xy^2 + xz^2}$$

Take $= \frac{xdy + ydy + zdz}{x(x^2 + y^2 + z^2)} = \frac{dy}{2xy}$

$$= \frac{2(xdx + ydy + zdz)}{x^2 + y^2 + z^2} = \frac{dy}{y}$$

Integrating $\text{Log}(x^2 + y^2 + z^2) = \text{Log } Y + \text{Log } C_1$.

(i.e.) $\frac{x^2 + y^2 + z^2}{y} = C_1$.

Again taking $\frac{dy}{2xy} = \frac{dz}{2xz}$ we get

$$\frac{dy}{y} = \frac{dz}{z}$$

$\text{Log } y = \text{Log } z + \text{Log } C_1$.

$$\frac{y}{z} = C_1$$

The solution is $\phi \left(\frac{x^2 + y^2 + z^2}{y}, \frac{y}{z} \right) = 0$

4) Prove that the general primitive of the equation $xp - yq = 2xe^{-(x^2+y^2)}$ can be expressed in the form.

$$Z = e^{2xy} \int_0^{x+y} e^{-u^2} du + e^{-2xy} \int_0^{x-y} e^{-u^2} du + F(x, y)$$

The equation $xp - yq = 2xe^{-(x^2+y^2)}$ is of the form $Pp + Qq = R$ whose Auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{2xe^{-(x^2+y^2)}}$$

From the first two

$$\log x = -\log y + \log c_1.$$

$$\therefore xy = c_1 \rightarrow (1)$$

From the first and the last ratios.

$$\frac{dx}{x} = \frac{dz}{2xe^{-(x^2+y^2)}}$$

$$(i.e.) \quad 2dx = \frac{dz}{e^{-(x^2+y^2)}}$$

$$dx + dy + dx - dy = \frac{dz}{e^{-(x^2+y^2)}}$$

$$\{d(x+y) + d(x-y)\} e^{-(x^2+y^2)} = dz.$$

$$e^{-(x^2+y^2)} d(x+y) + e^{-(x^2+y^2)} d(x-y) = dz.$$

$$e^{2xy} e^{-(x+y)^2} d(x+y) + e^{-2xy} e^{-(x-y)^2} d(x-y) = dz.$$

$$e^{2c_1} e^{-(x+y)^2} d(x+y) + e^{-2c_1} e^{-(x-y)^2} d(x-y) = dz.$$

$$\text{Integrating } Z = e^{2c_1} \int_0^{x+y} e^{-u^2} du + e^{-2c_1} \int_0^{x-y} e^{-u^2} du + C_2.$$

$$= e^{2c_1} \int_0^{x+y} e^{-u^2} du + e^{-2c_1} \int_0^{x-y} e^{-u^2} du + f(x,y)$$

Where C_2 is a function of x and y .

Exercises :

1. $x^2p + y^2q = z^2.$
2. $P + 3q = 5z + \tan(y-3x)$
3. $x(y^2-z^2)P + y(z^2-x^2)q = z(x^2-y^2)$
4. $(mz - ny)P + (nx - lz)q = ly - mx.$

8.3 Standard Forms :

Special methods of solving a p.d.e. in certain standard cases :

Standard : 1

Equations involving P and q only belong to this type. The equation can be put in the form $f(p, q) = c \rightarrow (1)$

An integral in this case is given by

$$Z = ax + by + c \rightarrow (2)$$

$$\therefore P = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = b$$

\therefore a and b satisfy the condition $f(a, b) = 0$. From this we can solve for b in terms of a, say $b = g(a)$.

$$\therefore (2) \text{ becomes } z = ax + yg(a) + c \rightarrow (3)$$

(3) is the complete solution of the given equation since it contains two arbitrary constants a and c. Having found the complete integral we can find the general integral by putting $c = \phi(a)$ and eliminating a between the equations.

$$Z = ax + yg(a) + \phi(a)$$

$$\text{and } \frac{\partial z}{\partial a} = 0$$

$$\therefore 0 = a + yg'(a) + \phi'(a)$$

For finding the singular integral we must eliminate a and c from the equations.

$$Z = ax + yg(a) + c$$

$$\frac{\partial z}{\partial a} = 0 \text{ and } \frac{\partial z}{\partial c} = 0$$

But in this case there is no singular solution. Since

$$\frac{\partial z}{\partial c} = 0 \Rightarrow 0 = 1$$

Example :

$$\text{Solve } Pq = k$$

Let the solution be $z = ax + by + c$ where $ab = k$.

$$\therefore b = \frac{k}{a}$$

∴ The complete integral is $z = ax + \frac{k}{a} y + c$

Differentiating partially w.r.to c, we see that $0=1$ which is absurd.

∴ There is no singular integral

To find the general integral, put $c = \phi(a)$.

$$\therefore Z = ax + \frac{k}{a} y + \phi(a)$$

Differentiating partially w.r.to 'a'

$$0 = x - \frac{k}{a^2} y + \phi'(a)$$

The eliminant of 'a' between the equations is the general integral.

Example :

$$\text{Solve } (x+y)(P+q)^2 + (x-y)(p-q)^2 = 1$$

$$\text{Put } x+y = x^2 \text{ and } x-y = y^2.$$

$$\begin{aligned} \therefore P &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial x} \\ &= \frac{\partial z}{\partial x} \frac{1}{2x} + \frac{\partial z}{\partial y} \frac{1}{2y} \end{aligned}$$

$$\begin{aligned} \text{Similarly } q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial y} \\ &= \frac{\partial z}{\partial y} \frac{1}{2x} - \frac{\partial z}{\partial y} \frac{1}{2y} \end{aligned}$$

$$\therefore P + q = \frac{1}{x} \frac{\partial z}{\partial x} \text{ and } P - q = \frac{1}{y} \frac{\partial z}{\partial y}$$

Substituting the equation becomes

$$x^2 \frac{1}{x^2} \left(\frac{\partial z}{\partial x} \right)^2 + y^2 \frac{1}{y^2} \left(\frac{\partial z}{\partial y} \right)^2 = 1$$

$$\text{(i.e.) } P^2 + Q^2 = 1 \text{ where } P = \frac{\partial z}{\partial x}; Q = \frac{\partial z}{\partial y}$$

∴ The solution is $Z = ax + by + c$ where

$$a^2 + b^2 = 1 \quad \therefore b = \sqrt{1-a^2}$$

$$\therefore Z = ax + \sqrt{1-a^2} y + c$$

$$(i.e.) Z = a\sqrt{x+y} + \sqrt{1-a^2}\sqrt{x-y} + c$$

Exercises :

Solve :

$$1. P^2 + q^2 = npq$$

$$2. P^2 - q^2 = 1$$

$$3. P^2 + q^2 = m^2.$$

Standard : 2

Equation of the form $Z = Px + qy + f(p, q) \rightarrow (1)$ belong to this type.

Consider the relation $Z = ax + by + f(a, b) \rightarrow (2)$ where a and b are constants.

$$P = \frac{\partial z}{\partial x} = a \quad q = \frac{\partial z}{\partial y} = b.$$

∴ (2) is obtained by putting $p=a$ and $q=b$ in $\rightarrow (1)$

∴ The complete integral of $Z = Px + qy + f(p, q)$ is p and q by a and b respectively.

Hence this type is analogous to Clairaut's form in unit 1. For getting the general solution we replace b by $\phi(a)$ so that $z = ax + \phi(a)y + f[a, \phi(a)] \rightarrow (3)$

$$\text{and } \frac{\partial z}{\partial a} = x + \phi'(a)y + f'(a) = 0 \rightarrow (4)$$

The eliminant of ' a ' between (3) and (4) is the general solution.

In order to obtain the singular solution, we eliminate ' a ' and ' b ' between the equations $z = ax + by + f(a, b)$

$$\frac{dz}{da} = 0 \quad \text{and} \quad \frac{\partial z}{\partial b} = 0$$

Example :

$$\text{Solve } Z = Px + qy + P^2 + q^2.$$

This is of the form $Z = Px + qy + f(P, q)$

The complete integral is $Z = ax + by + f(a, b)$

$$\therefore Z = ax + by + a^2 + b^2$$

Differentiating partially w.r.to 'a' and 'b'.

We get $0 = x + 2a$ and $0 = y + 2b$.

$$\therefore a = -\frac{x}{2} \text{ and } b = -\frac{y}{2}$$

$$\therefore (1) \text{ becomes } Z = \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4}$$

2. Solve $Z = px + qy + c\sqrt{1+p^2+q^2}$

This is of the form $z = px + qy + f(p, q)$

\therefore The complete integral is $z = ax + by + f(a, b)$

$$(i.e.) Z = ax + by + c\sqrt{1+a^2+b^2} \rightarrow (1)$$

Differentiating partially w.r. to 'a' and 'b' we get

$$0 = x + c \frac{a}{\sqrt{1+a^2+b^2}} \rightarrow (2)$$

$$0 = y + c \frac{b}{\sqrt{1+a^2+b^2}} \rightarrow (3)$$

$$\therefore x^2 + y^2 = \frac{c^2(a^2 + b^2)}{1 + a^2 + b^2}$$

$$\therefore c^2 - x^2 - y^2 = C^2 - \frac{c^2(a^2 + b^2)}{1 + a^2 + b^2} = \frac{c^2}{1 + a^2 + b^2}$$

$$\therefore \sqrt{1 + a^2 + b^2} = \frac{C}{\sqrt{c^2 - x^2 - y^2}}$$

$$\therefore (2) \Rightarrow a = \frac{-x}{\sqrt{c^2 - x^2 - y^2}}$$

$$\therefore (3) \Rightarrow b = \frac{-y}{\sqrt{c^2 - x^2 - y^2}}$$

$$\therefore (1) \Rightarrow z = \frac{-x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + C \frac{1}{\sqrt{c^2-x^2-y^2}}$$

$$= \frac{c^2-x^2-y^2}{\sqrt{c^2-x^2-y^2}} = \sqrt{c^2-x^2-y^2}$$

$$\therefore Z^2 = c^2-x^2-y^2$$

$$x^2 + y^2 + z^2 = c^2.$$

Exercises :

$$1. Z = px + qy + pq$$

$$2. z = px + qy - 2$$

$$3. Z = px + qy + \sqrt{ap^2 + aq^2 + a}.$$

Standard : 3

Equation of the form $f(z, p, q) = 0$ which do not contain x and y belong to this type.

Let $z = f(x)$ where $x = x + ay \rightarrow (1)$ be a tentative solution.

$$\therefore P = \frac{\partial z}{\partial x} = \frac{dz}{dx} \frac{\partial x}{\partial x} = \frac{dz}{dx} \cdot 1 = \frac{dy}{dx}$$

$$Q = \frac{\partial z}{\partial y} = \frac{dz}{dx} \frac{\partial x}{\partial y} = a \frac{dz}{dx}$$

$$\therefore f \left[Z, \frac{dz}{dx}, a \frac{dz}{dx} \right] = 0 \rightarrow (2)$$

(2) is an ordinary differential equation of the first order.

$$\text{We obtain a relation } \frac{dz}{dx} = g(z, a)$$

$$\text{Seperarting } \frac{dz}{g(z, a)} = dx.$$

$$\text{Integrating } h(z, a) = x + c.$$

$$(\text{i.e.}) x + ay + c = h(z, a)$$

Which is the complete integral. The general and particular integrals can be obtained as usual.

Example :

Solve $p^3 + q^3 = 27z$.

The equation does not contain x and y .

Put $x = x + ay$.

$$P = \frac{dz}{dx} \text{ and } q = a \frac{dz}{dx}$$

\therefore The given equation becomes

$$\left(\frac{dz}{dx} \right)^3 + a^3 \left(\frac{dz}{dx} \right)^3 = 27z.$$

$$\left(\frac{dz}{dx} \right)^3 (1 + a^3) = 27z.$$

$$\frac{dz}{dx} + \frac{3z^{1/3}}{(1+a^3)^{1/3}}$$

$$(i.e.) \frac{(1+a^3)^{1/3}}{3} \frac{dz}{z^{1/3}} = dx$$

$$\text{Integrating } \frac{(1+a^3)^{1/3}}{3} \frac{z^{2/3}}{2/3} = x + b.$$

$$\text{cubing } \frac{1+a^3}{8} z^2 = (x+b)^3.$$

$$(i.e.) (1+a^3) z^2 = 8 (x+ay+b)^3.$$

Which is the complete integral.

Example :

Solve $q^2 y^2 = z (z - px)$

Put $\text{Log } x = u$ and $\text{Log } y = v$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{dx}{du} = xp$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial y} \frac{dy}{dv} = yq$$

∴ The given equation becomes.

$$\left(\frac{\partial z}{\partial v} \right)^2 = Z \left[Z - \frac{\partial z}{\partial u} \right]$$

(i.e.) $q_1^2 = Z(Z - p_1)$ where $\frac{\partial z}{\partial v} = q_1$ and $\frac{\partial z}{\partial u} = p_1$.

Now this is of the form $f(z, p_1, q_1) = 0$.

Let $x = u + av$ be a tentative solution.

$$\frac{\partial z}{\partial u} = \frac{dz}{dx} \text{ and } \frac{\partial z}{\partial v} = a \frac{dz}{dx}$$

$$\therefore a^2 \left(\frac{dz}{dx} \right)^2 + z \left(\frac{dz}{dx} \right) - z^2 = 0$$

$$\frac{dz}{dx} = \frac{-z \pm \sqrt{z^2 + 4a^2 z^2}}{2a^2}$$

$$= \frac{z}{2a^2} \left[-1 \pm \sqrt{1 + 4a^2} \right]$$

(i.e.) $2a^2 \frac{dz}{z} = \left[-1 \pm \sqrt{1 + 4a^2} \right] dx$

Integrating $\frac{2a^2}{-1 \pm \sqrt{1 + 4a^2}} \text{Log } z = x + b$

$$\text{Log } Z = \frac{-1 \pm \sqrt{1 + 4a^2}}{2a^2} (u + av + b)$$

Where $u = \text{Log } x$; $V = \text{Log } y$.

Exercises :

1. $q^2 + z^2 p^4 = z^2 p^2$.

2. $z^2 (p^2 + q^2 + 1) = a^2$.

3. $P(1 + q) = qz$.

Standard : 4

Equations of the form $f_1(x, p) = f_2(y, q)$ belong to this type.

$$\text{Let } f_1(x, p) = f_2(y, q) = k$$

Taking $f_1(x, p) = k$ we get $p = f_1(x, k)$

$$f_2(y, q) = k \text{ gives } q = f_2(y, k)$$

On integration (1) and (2) lead to

$$Z = \int F_1(x, k) dx + \text{a quantity independent of } x.$$

$$Z = \int F_2(y, k) dy + \text{a quantity independent of } y.$$

\therefore The complete integral will be given by

$$Z = \int F_1(x, k) dx + \int F_2(y, k) dy + b.$$

Note :

The general integral can be found out as usual.

Example :

1) Solve. $q - p + x - y = 0$

The given equation becomes $p - x = q - y$.

$$\text{Let } p - x = q - y = k.$$

$$\therefore p - x = k \text{ and } q - y = k.$$

$$\text{(i.e.) } p = x + k \text{ and } q = y + k.$$

$$\text{Integrating } Z = \frac{x^2}{2} + kx + c_1.$$

$$Z = \frac{y^2}{2} + ky + c_2.$$

\therefore The complete integral is

$$Z = \frac{x^2 + y^2}{2} + k(x + y) + c_1 + c_2.$$

$$\text{(i.e.) } 2z = x^2 + y^2 + 2k(x + y) + b$$

The partial differentiation wr. t. 'b' leads to an absurd result $0 = 1$.

There is no singular integral to find the general integral let $b = f(k)$ where f is arbitrary.

$$\therefore 2z = x^2 + y^2 + 2k(x+y) + f(k) \rightarrow (1)$$

Differentiating w.r. to k partially,

$$0 = 2(x+y) + f'(k) \rightarrow (2)$$

Eliminating k between (1) and (2) we get the general integral.

2) Solve $q(p - \sin x) = \cos y$

$$\therefore p - \sin x = \frac{\cos y}{q} = k$$

$$\therefore p = \sin x + k \text{ and } q = \frac{\cos y}{k}$$

$$\text{Integrating } Z = -\cos x + kx + c_1.$$

$$\text{and } Z = \frac{\sin y}{k} + c_2$$

$$\text{The complete integral is } Z = -\cos x + kx + \frac{\sin y}{k} + b.$$

\therefore The partial differentiation w.r. to ' b ' leads to an absurd result there is no singular integral.

For the general integral assume $b = f(k)$

$$\therefore Z = -\cos x + kx + \frac{\sin y}{k} + f(k) \rightarrow (2).$$

Differentiating partially w.r. to ' k '.

$$0 = x - \frac{\sin y}{k^2} + f'(k) \rightarrow (2)$$

Eliminating k between (1) and (2) we get the general integral.

Exercises : Solve :

1. $p^2 + q^2 = x + y$

2. $\sqrt{p} + \sqrt{q} = 2x$

3. $p - 3x^2 = q^2 - y.$

4. $Pq = xy.$

UNIT - 10

10.1 The Laplace Transforms :

The Laplace transforms has been widely adopted by scientists and engineers as an efficient tool for solving linear differential equations. In this chapter we shall develop the fundamental properties of Laplace transforms and we shall see how they are used to solve linear differential equations and certain other problems.

Definition :

If a function $f(t)$ is defined for all positive values of the variable t and if $\int_0^{\infty} e^{-st} f(t) dt$ exists and is equal to $F(s)$, then $F(s)$ is called the Laplace transform of $f(t)$ and is denoted by the symbol $L\{f(t)\}$.

Hence $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = F(s)$. The operator L that transform $f(t)$ into $F(s)$ is called the Laplace transform operator.

Note : Lt

$$S \rightarrow \infty F(s) = 0$$

Definitions : Piecewise Continuity

A function $f(t)$ is said to be piecewise continuous in a closed interval $[ab]$ if it is defined on that interval and is such that the interval can be broken up into a finite number of sub intervals in each of which $f(t)$ is continuous. $f(t)$ can be have only ordinary finite discontinuities in the interval.

Exponential Order :

A function $f(t)$ is said to be of exponential order if $\lim_{t \rightarrow \infty} e^{-st} f(t) = 0$ (or) for some number S_0 , the product $e^{-s_0 t} |f(t)| < m$ for $t > T$ (ie) $e^{-s_0 t} |f(t)|$ is bounded for large values of t , say for $t > T$.

Sufficient conditions for the existence of the Laplace transform :

- (i) $f(t)$ is continuous (or) piecewise continuous in the closed interval $[ab]$ where $a > 0$.
- (ii) It is of exponential order.
- (iii) $t^n f(t)$ is bounded near $t=0$ for some number $n > 1$.

From the definition the following results can easily be proved :

$$(i) \quad L\{f(t) + \phi(t)\} = L\{f(t)\} + L\{\phi(t)\}$$

$$\text{We have } L\{f(t) + \phi(t)\} = \int_0^{\infty} e^{-st} \{f(t) + \phi(t)\} dt$$

$$= \int_0^{\infty} e^{-st} \{f(t) dt + \int_0^{\infty} e^{-st} \phi(t) dt$$

$$= [L \{f(t)\} + L \{\phi(t)\}]$$

(ii) $L \{cf(t)\} = cL \{f(t)\}$ where C is a constant

$$\text{We have } L \{cf(t)\} = \int_0^{\infty} e^{-st} cf(t) dt$$

$$= C \int_0^{\infty} e^{-st} f(t) dt$$

$$= cL \{f(t)\}$$

(iii) $L \{f^1(t)\} = sL \{f(t)\} - f(0)$

$$\text{We have } L \{f^1(t)\} = \int_0^{\infty} e^{-st} f^1(t) dt$$

$$= f(t) [e^{-st}]_0^{\infty} - \int_0^{\infty} f(t) (-s)e^{-st} dt$$

$$= -f(0) + S \int_0^{\infty} f(t) e^{-st} dt$$

$$= sL \{f(t)\} - f(0)$$

(iv) $L \{f^1(t)\} = S^2 L \{f(t)\} - sf(0) - f^1(0)$

$$L \{f^1(t)\} = L \{f^1(t)\} \text{ where } F(t) = f^1(t)$$

$$= SL \{f(t)\} - f(0)$$

$$= SL \{f^1(t)\} - f^1(0)$$

$$= S [SL \{f(t)\} - f(0)] - f^1(0)$$

$$= S^2 L \{f(t)\} - sf(0) - f^1(0)$$

(v) By extending the previous result, we get

$$L \{f^n(t)\} = S^n L \{f(t)\} - S^{n-1} f(0)$$

$$- S^{n-2} f^1(0) \dots \dots f^{n-1}(0)$$

(vi) If $L \{f(t)\} = F(s)$

$$\begin{matrix} L_t & L_t \\ (a) & t \rightarrow 0 \quad f(t) = S \rightarrow \infty SF(s) \end{matrix}$$

$$\begin{matrix} L_t & L_t \\ (b) & t \rightarrow \infty \quad f(t) = S \rightarrow 0 SF(s) \end{matrix}$$

$$L \{f'(t)\} = SF(s) - f(0)$$

$$= SF(s) - f(0)$$

Taking Limit as $s \rightarrow \infty$ on both sides, we get

$$\begin{aligned} \lim_{s \rightarrow \infty} [SF(s) - f(0)] &= \lim_{s \rightarrow \infty} L\{f'(t)\} \\ &= \lim_{s \rightarrow \infty} \int_0^{\infty} e^{-st} f'(t) dt \\ &= 0 \end{aligned}$$

$$\therefore \lim_{s \rightarrow \infty} SF(s) = f(0)$$

$$\lim_{t \rightarrow 0} f(t)$$

This result is known as Initial value theorem. Taking limit as $s \rightarrow 0$ on both sides of $L \{f'(t)\}$, we get

$$\begin{aligned} \lim_{s \rightarrow 0} [SF(s) - f(0)] &= \lim_{s \rightarrow 0} \int_0^{\infty} e^{-st} f'(t) dt \\ &= \int_0^{\infty} f'(t) dt \\ &= [f(t)]_0^{\infty} \\ &= \lim_{t \rightarrow \infty} f(t) - f(0) \end{aligned}$$

$$\therefore \lim_{s \rightarrow 0} SF(s) = \lim_{t \rightarrow \infty} f(t)$$

This result is known as final value theorem.

$$(viii) \quad L(e^{-at}) = \frac{1}{s+a} \text{ provided } s+a > 0$$

$$\begin{aligned} L(e^{-at}) &= \int_0^{\infty} e^{-st} e^{-at} dt \\ &= \int_0^{\infty} e^{-(s+a)t} dt \end{aligned}$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty}$$

$$= \frac{1}{(s+a)}$$

Similarly $L(e^{at}) = \frac{1}{s-a}$ provided $s-a > 0$

Cor : $L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right)$

$$= \frac{1}{2} L(e^{at}) + \frac{1}{2} L(e^{-at})$$

$$= \frac{1}{2} \frac{1}{s-a} + \frac{1}{2} \frac{1}{s+a}$$

$$= \frac{s}{s^2 - a^2}$$

Similarly $L(\sinh at) = \frac{a}{s^2 - a^2}$

(Viii) $L(\cos at) = \frac{s}{s^2 + a^2}$

Pf : $L(\cos at) = \int_0^{\infty} e^{-st} \cos at \, dt$

$$= \left[e^{-st} \frac{-s \cos at + a \sin at}{s^2 + a^2} \right]_0^{\infty}$$

$$= \frac{s}{s^2 + a^2}$$

(ix) $L(\sin at) = \frac{a}{s^2 + a^2}$

$$L(\sin at) = \int_0^{\infty} e^{-st} \sin at \, dt$$

$$= \left[e^{-st} \frac{-s(\sin at) - a \cos at}{s^2 + a^2} \right]_0^{\infty}$$

$$= \frac{a}{s^2 + a^2}$$

$$(x) L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\text{We have } L(t^n) = \int_0^{\infty} e^{-st} t^n dt$$

$$\text{Put } st = x \text{ then } dt = \frac{1}{s} dx$$

$$\therefore L(t^n) = \int_0^{\infty} \left(\frac{x}{s} \right)^n e^{-x} \frac{1}{s} dx$$

$$= \frac{1}{s^{n+1}} \int_0^{\infty} x^n e^{-x} dx$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

when n is a positive integer $\Gamma(n+1) = n!$

$$L(t^n) = \frac{n!}{s^{n+1}} \text{ when } n \text{ is a +ve integer.}$$

$$\text{Cor :- } L(1) = \frac{1}{s} \quad L(t) = \frac{1}{s^2} \quad L(t^2) = \frac{2}{s^3}$$

$$L(t^{1/2}) = \frac{\Gamma\left\{\frac{3}{2}\right\}}{s^{3/2}} = \frac{\frac{1}{2} \sqrt{(1/2)}}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$L(t^{-1/2}) = \frac{(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$$

Examples :

1) Find $L(t^2 + 2t + 3)$

$$L(t^2 + 2t + 3) = L(t^2) + L(2t) + 3L(1)$$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}$$

:

2) Find $L(\sin^3 2t)$

$$\text{Since } \sin 6t = 3\sin 2t - 4\sin^3 2t$$

$$\begin{aligned} \text{We have } L(\sin^3 2t) &= L\left[\frac{3\sin 2t - \sin 6t}{4}\right] \\ &= \frac{3}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t) \\ &= \frac{3}{4} \cdot \frac{2}{S^2 + 2^2} - \frac{1}{4} \cdot \frac{6}{S^2 + 6^2} \\ &= \frac{3}{48} - \frac{6}{4(S^2 + 36)} \\ &= \frac{3(S^2 + 36) - 48}{48(S^2 + 36)} \end{aligned}$$

3) Find $L\{f(t)\}$, where

$$f(t) = 0 \text{ when } 0 < t < 2$$

$$= 3 \text{ when } t > 2$$

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt \\ &= \int_0^2 e^{-st} f(t) dt + \int_2^\infty e^{-st} f(t) dt \\ &= \int_0^2 e^{-st} (0) dt + \int_2^\infty e^{-st} (3) dt \\ &= 3 \int_2^\infty e^{-st} dt \\ &= \frac{3}{S} e^{-2s} \end{aligned}$$

Laplace transform of Periodic Functions :

Let $f(t)$ be a function with period 'a'. Then $f(t) = f(a+t) = f(2a+t) = \dots = f(na+t)$.

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt \\ &= \int_0^a e^{-st} f(t) dt + \int_a^{2a} e^{-st} f(t) dt \\ &\quad + \int_{2a}^{3a} e^{-st} f(t) dt + \dots + \int_{(n-1)a}^{na} e^{-st} f(t) dt \end{aligned}$$

In the second integral put $t = T + a$

In the third integral put $t = T + 2a$

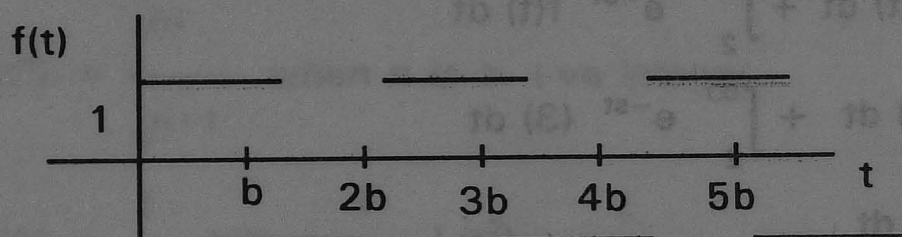
In the fourth integral put $t = T + 3a$

In the n^{th} integral put $t = T + (n-1)a$ and so on.

$$\begin{aligned}
 \text{Hence } L\{f(t)\} &= \int_0^a e^{-st} f(t) dt + \int_0^a e^{-s(T+a)} f(T+a) dT \\
 &+ \int_0^a e^{-s(T+2a)} f(T+2a) dT + \dots \\
 &= \int_0^a e^{-st} f(t) dt + e^{-sa} \int_0^a e^{-st} f(t) dt + e^{-2sa} \int_0^a e^{-st} f(t) dt + \dots \\
 &= (1 + e^{-sa} + e^{-2sa} + \dots) \int_0^a e^{-st} f(t) dt \\
 &= \frac{1}{1 - e^{-sa}} \int_0^a e^{-st} f(t) dt
 \end{aligned}$$

Examples :

1) Find the transform of the rectangular wave shown below :



Here $f(t) = 1$ when $0 < t < b$

$= -1$ when $b < t < 2b$

The function is periodic in the interval $(0, 2b)$

$$\text{Hence } f(t) = \frac{1}{1 - e^{-2bs}} \int_0^{2b} e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{-2bs}} \left(\int_0^b e^{-st} f(t) dt + \int_b^{2b} e^{-st} f(t) dt \right)$$

$$= \frac{1}{1 - e^{-2bs}} \left(\int_0^b e^{-st} f(t) dt + \int_b^{2b} e^{-st} (-1) dt \right)$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \left(\frac{e^{-st}}{-s} \right) + \left(\frac{e^{-st}}{s} \right) \right\}$$

$$= \frac{1}{S} \left(\frac{1-e^{-2sb} + e^{-2sb}}{1-e^{-2sb}} \right)$$

$$= \frac{1}{S} \left(\frac{1-e^{-sb}}{1+e^{-sb}} \right)$$

$$= \frac{1}{S} \tanh \left(\frac{bs}{2} \right)$$

Some general theorems :

(i) If $L\{f(t)\} = F(s)$, then $L\{f(at)\} = \frac{1}{a} F\left(\frac{S}{a}\right)$

$$L\{f(at)\} = \int_0^{\infty} e^{-st} f(at) dt \text{ put } at=y$$

$$= \frac{1}{a} \int_0^{\infty} e^{-\frac{S}{a} y} f(y) dy$$

$$= \frac{1}{a} F\left(\frac{S}{a}\right)$$

For example :

1) $L(\cos at) = \frac{1}{a} F\left(\frac{S}{a}\right)$ where

$$L(\cos t) = F(s) = \frac{S}{S^2+1}$$

$$L(\cos C at) = \frac{1}{a} \frac{\frac{S/a}{S^2 + 1}}{\frac{S/a}{a^2} + 1} = \frac{S}{S^2 + a^2}$$

2) $L(\sin at) = \frac{1}{a} F\left(\frac{S}{a}\right)$ where

$$L(\sin t) = F(s) = \frac{1}{S^2-1}$$

$$L(\sinh(at)) = \frac{1}{a} \cdot \frac{1}{\frac{S^2}{a^2} - 1} = \frac{a}{S^2 - a^2}$$

(ii) $L\{e^{-at} f(t)\} = F(s+a)$ where $F(s) = L\{f(t)\}$

$$\begin{aligned} \text{We have } L\{e^{-at} f(t)\} &= \int_0^{\infty} e^{-st} e^{-at} f(t) dt \\ &= \int_0^{\infty} e^{-(s+a)t} f(t) dt \end{aligned}$$

This is in structure exactly the Laplace transform of $f(t)$ itself except that $s+a$ takes the place of s .

Examples :

$$1) L(1) = \frac{1}{S} \quad \therefore L(e^{-at}) = \frac{1}{S+a}$$

$$2) L(\cos bt) = \frac{S}{S^2 + b^2}$$

$$L(e^{-at} \cos bt) = \frac{S+a}{(S+a)^2 + b^2}$$

$$3) L(\sin bt) = \frac{b}{S^2 + b^2}$$

$$L(e^{-at} \sin bt) = \frac{b}{(S+a)^2 + b^2}$$

(iii) If $L\{f(t)\} = F(s)$. Then $L\{tf(t)\} = -\frac{d}{ds} F(s)$

$$F(s) = \int_0^{\infty} e^{-St} f(t) dt$$

$$\begin{aligned} \therefore \frac{d}{ds} F(s) &= \frac{d}{ds} \int_0^{\infty} e^{-St} f(t) dt \\ &= \int_0^{\infty} \frac{\delta}{\delta S} \left(e^{-St} f(t) \right) dt \end{aligned}$$

$$= \int_0^{\infty} -t e^{-St} f(t) dt$$

$$= - \int_0^{\infty} e^{-st} t f(t) dt$$

$$= - L \{ t f(t) \}$$

$$\therefore L \{ f(t) \} = \frac{-d}{ds} F(s)$$

$$\text{Cor : } L \{ t^n f(t) \} = (-1)^n \frac{d^n}{ds^n} \{ L(f(t)) \}$$

$$\text{We have } L \{ t f(t) \} = - \frac{d}{ds} L(f(t))$$

$$L \{ t^2 f(t) \} = L \{ t \cdot t f(t) \}$$

$$= - \frac{d}{ds} L \{ t f(t) \}$$

$$= - \frac{d}{ds} \left(- \frac{d}{ds} L \{ f(t) \} \right)$$

$$= (-1)^2 \frac{d^2}{ds^2} \{ L(f(t)) \}$$

Continuing this process, we get the result. This result can also be written as follows.

If $L \{ f(t) \} = F(s)$ Then

$$F'(s) = L \{ -t f(t) \}$$

$$F''(s) = L \{ (-t)^2 f(t) \}$$

$$F^{(n)}(s) = L \{ (-t)^n f(t) \}$$

Examples :

1) Find $L \{ t e^{-at} \}$

$$L \{ t e^{-at} \} = - \frac{d}{ds} L \{ e^{-at} \}$$

$$= - \frac{d}{ds} \left(\frac{1}{s+a} \right) = \frac{1}{(s+a)^2}$$

2) Find $L \left[\frac{\sin at}{t} \right]$

$$\left[\frac{L(\sin at)}{t} \right] = \int_0^\infty L(\sin at) ds. \text{ Since } \lim_{t \rightarrow 0} \frac{\sin at}{t} = a$$

$$= \int_0^\infty \frac{a}{s^2 + a^2} ds$$

$$= \left[\tan^{-1} \frac{s}{a} \right]_0^\infty$$

$$= \tan^{-1} \infty - \tan^{-1} \frac{0}{a}$$

$$= \frac{\pi}{2} - \tan^{-1} \frac{0}{a}$$

$$= \cot^{-1} \frac{0}{a}$$

10.2 Inverse Laplace Transforms :

Let the symbol $L^{-1} \{f(s)\}$ denote the function whose Laplace transform is $F(s)$.

Thus if $L \{f(t)\} = F(s)$ then $F(t) = L^{-1}\{F(s)\}$. The most obvious way of finding the inverse transform of a given function is to look into the table of transform and get the function whose Laplace transform is the given function.

We can compile the table of transforms from the known results.

1.	$F(t)$	$F(s)$
2.	e^{at}	$\frac{1}{s-a}$
3.	$\cosh(at)$	$\frac{1}{s^2 - a^2}$
4.	$\sinh at$	$\frac{a}{s^2 - a^2}$
5.	$\cos at$	$\frac{s}{s^2 + a^2}$
6.	1	$\frac{1}{s}$
	$\sin at$	$\frac{a}{s^2 + a^2}$

7. t	$\frac{1}{s^2}$
8. t^n	$\frac{n!}{s^{n+1}}$ (n is a five integer)
9. te^{at}	$\frac{1}{(s-a)^2}$
10. t^2e^{at}	$\frac{2}{(s-a)^3}$
11. t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$ (n is a five integer)
12. $e^{-at} \sin bt$	$\frac{b}{(s+a)^2 + b^2}$
13. $e^{-at} \cos bt$	$\frac{s+a}{(s+a)^2 + b^2}$
14. $t \sin at$	$\frac{2as}{(s^2 + a^2)^2}$
15. $t \cos at$	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$

we can modify the results we have obtained in finding the Laplace transforms of functions to get the inverse transforms of functions.

(i) If $L\{f(t)\} = F(s)$ Then $L\{e^{-at} f(t)\} = F(s+a)$ Hence we get the result.

$L^{-1}[F(s+a)] = e^{-at} f(t) = e^{-at} L^{-1} F(s)$ Thus for example.

$$1) \quad L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at} L^{-1}\left[\frac{1}{s^2}\right] = e^{-at} t$$

$$2) \quad L^{-1}\frac{1}{(s+2)^2 + 16} = e^{-at} L^{-1}\frac{1}{s^2 + 4^2} = \frac{e^{-2t} \sin 4t}{4}$$

$$3) \quad L^{-1}\frac{s-3}{(s-2)^2 + 4} = e^{3t} L^{-1}\left[\frac{s}{s^2 + 4}\right] = e^{3t} \cos 2t$$

$$4) \quad L^{-1} \frac{S}{S^2 + 2s + 5} = L^{-1} \frac{S}{(S+1)^2 + 2^2}$$

$$= L^{-1} \frac{(S+1) - 1}{(S+1)^2 + 2^2}$$

$$= L^{-1} \frac{(S+1)}{(S+1)^2 + 2^2} - L^{-1} \left(\frac{1}{(S+1)^2 + 2^2} \right)$$

$$= e^{-t} L^{-1} \left(\frac{S}{S^2 + 2^2} \right) - e^{-t} L^{-1} \left(\frac{1}{S^2 + 2^2} \right)$$

$$= e^{-t} \cos 2t - e^{-t} \frac{\sin 2t}{2}$$

$$= \frac{e^{-t}}{2} (2 \cos 2t - \sin 2t)$$

$$(ii) \text{ If } L \{f(t)\} = F(s) \text{ then } L(f(at)) = \frac{1}{a} F\left(\frac{S}{a}\right)$$

This result can be written in the form.

$$L^{-1} \left(\frac{1}{a} F \left(\frac{S}{a} \right) \right) = f(at) \text{ where } f(t) = L^{-1} F(s)$$

Putting $\frac{1}{a} = k$ we have

$$L^{-1} F(ks) = \frac{1}{k} f \left(\frac{t}{k} \right) \text{ where } f(t) = L^{-1} F(s)$$

Example :-

1) Find $L^{-1} \left[\frac{S}{(S^2 + a^2)^2} \right]$

$$F'(S) = \frac{S}{(S^2 + a^2)^2}$$

$$F(S) = \int \frac{SdS}{(S^2 + a^2)^2}$$

$$= - \frac{1}{2(S^2 + a^2)}$$

$$\therefore L^{-1} \left[\frac{S}{(S^2 + a^2)^2} \right] = -t L^{-1} \left[\frac{1}{2(S^2 + a^2)} \right]$$

$$= \frac{t}{2} L^{-1} \left[\frac{1}{(S^2 + a^2)} \right]$$

$$= \frac{t}{2a} \sin at$$

2) Find $L^{-1} \left[\frac{S}{(S^2 - 1)^2} \right]$

$$\text{Here } F'(S) = \frac{S}{(S^2 - 1)^2}$$

$$F(S) = \int \frac{S}{(S^2 - 1)^2} ds.$$

$$= - \frac{1}{2(S^2 - 1)}$$

$$\therefore L^{-1} \left[\frac{S}{(S^2 - 1)^2} \right] = -t L^{-1} \left[\frac{1}{2(S^2 - 1)} \right]$$

$$= \frac{t}{2} L^{-1} \left[\frac{1}{S^2 - 1} \right]$$

$$= \frac{t}{2} \sinh t$$

$$(iii) \quad L \{f(t)\} = F(s) \text{ then } L \{t f(t)\} = -F'(s)$$

Hence we get the result.

$$\begin{aligned} L^{-1} \{F'(s)\} &= -t f(t) \\ &= -t L^{-1} \{F(s)\} \end{aligned}$$

Example :-

$$\text{Find } L^{-1} \left(\frac{S+2}{(S^2+4S+5)^2} \right)$$

$$\text{Here } F'(S) = \frac{S+2}{(S^2+4S+5)^2}$$

$$F(S) = -\frac{1}{2(S^2+4S+5)}$$

$$\therefore L^{-1} \left(\frac{S+2}{(S^2+4S+5)^2} \right) = -t L^{-1} \left(\frac{1}{-2(S^2+4S+5)} \right)$$

$$= \frac{t}{2} L^{-1} \left(\frac{1}{S^2+4S+5} \right)$$

$$= \frac{t}{2} L^{-1} \left(\frac{1}{(S+2)^2+1^2} \right)$$

$$= \frac{t}{2} e^{-2t} L^{-1} \left(\frac{1}{S^2+1^2} \right)$$

$$= \frac{t e^{-2t} \sin t}{2}$$

$$(iv) \quad \text{If } L \{f(t)\} = F(S) \text{ then } L \{t f(t)\} = -F'(S)$$

This theorem can be used in the following way to get inverse transforms of certain functions. Take for example

$$L^{-1} \left[\log \frac{S+1}{S-1} \right]$$

Let this be equal to $f(t)$.

$$\text{Then } L\{f(t)\} = \log \frac{S+1}{S-1}$$

$$L\{t f(t)\} = - \frac{d}{dS} \log \frac{S+1}{S-1}$$

$$L\{t f(t)\} = - \frac{d}{dS} \log \frac{S+1}{S-1}$$

$$= - \frac{d}{dS} \left[\log (S+1) - \log (S-1) \right]$$

$$= - \frac{1}{S+1} + \frac{+1}{S-1}$$

$$\therefore t f(t) = L^{-1} \left[\frac{1}{S-1} \right] - L^{-1} \left[\frac{1}{S+1} \right]$$

$$= e^t - e^{-t}$$

$$= 2 \sin (ht)$$

$$\therefore f(t) = \frac{2 \sin (ht)}{t}$$

(v) If $2 \{f(t)\} = S F(s)$ and $\phi(t)$ is a function such that $L[\phi(t)] = F(s)$ and $\phi(0) = 0$ then $f(t) = \phi'(t)$ we have $L[\phi'(t)] = S L(\phi(t)) - \phi(0)$

$$= S F(S)$$

$$= L[f(t)]$$

$$\therefore f(t) = \phi'(t)$$

This result can be used to get the inverse transforms of certain functions.

$$L^{-1} \{S F(S)\} = f(t)$$

$$= \frac{d}{dt} \phi(t)$$

$$= \frac{d}{dt} L^{-1} \{F'(S)\}$$

Provided $L^{-1} (F(S)) = 0$ when $t=0$.

Examples :-

1) Find $L^{-1} \left(\frac{S}{S^2 + k^2} \right)$

$$\begin{aligned} L^{-1} \left(\frac{S}{S^2 + k^2} \right) &= \frac{d}{dt} L^{-1} \left(\frac{1}{S^2 + k^2} \right) \\ &= \frac{d}{dt} \left(\frac{\sin kt}{k} \right) \\ &= \cos kt. \end{aligned}$$

Here $\frac{\sin kt}{k} = 0$. When $t = 0$

2) Find $L^{-1} \left(\frac{S^2}{(S-1)^3} \right)$

$$L^{-1} \left(\frac{S^2}{(S-1)^3} \right) = \frac{d}{dt} L^{-1} \left(\frac{S}{(S-1)^3} \right)$$

$$= \frac{d^2}{dt^2} L^{-1} \left(\frac{1}{(S-1)^3} \right)$$

$$= \frac{d^2}{dt^2} \left(\frac{e^t t^2}{2} \right)$$

$$= \frac{e^t}{2} (t^2 + 4t + 2)$$

$$(vi) \quad L \int_0^t f(x) dx = \frac{1}{S} L[f(t)]$$

$$\text{Let } \int_0^t f(x) dx \text{ be } F(t)$$

$$\text{Then } F'(t) = f(t) \text{ and } f(0) = 0.$$

$$\begin{aligned} L[F'(t)] &= S L[f(t)] - F(0) \\ &= S L[F(t)] \end{aligned}$$

$$(i.e.) \quad L\{f(t)\} = S L\left[\int_0^t f(x) dx\right]$$

$$\text{Hence } L \int_0^t f(x) dx = \frac{1}{S} L[f(t)]$$

This result can also be used to find the inverse transforms of certain functions.

$$\int_0^t f(x) dx = L^{-1} \left[\frac{1}{S} L[f(t)] \right]$$

$$\text{If } L\{f(t)\} = F(s)$$

$$\text{Then } L^{-1} \left[\frac{1}{S} F(s) \right] = \int_0^t f(x) dx$$

$$\text{Where } f(t) = L^{-1} F(s)$$

$$\therefore L^{-1} \left[\frac{1}{S} F(s) \right] = \int_0^t L^{-1} F(s) dt$$

Example :-

$$1) \quad \text{Find } L^{-1} \left[\frac{1}{S(S+a)} \right]$$

$$\begin{aligned} L^{-1} \left[\frac{1}{S(S+a)} \right] &= \int_0^t L^{-1} \left[\frac{1}{S+a} \right] dt \\ &= \int_0^t e^{-at} dt \end{aligned}$$

$$= \left[\frac{e^{-at}}{-a} \right]_0^t$$

$$= \frac{1}{a} \left[1 - e^{-at} \right]$$

(vii) The method of partial fractions can be used to find the inverse transform of certain functions. The method is illustrated in the following examples.

Example :-

1) Find $L^{-1} \left[\frac{1}{S(S+1)(S+2)} \right]$

We can split $\frac{1}{S(S+1)(S+2)}$ into partial fraction as

$$\frac{1}{2} - \frac{1}{S} = \frac{1}{S+1} + \frac{1}{2} - \frac{1}{S+2}$$

$$L^{-1} \left[\frac{1}{S(S+1)(S+2)} \right] = \frac{1}{2} L^{-1} \left[\frac{1}{S} \right] - L^{-1} \left[\frac{1}{S+1} \right]$$

$$+ \frac{1}{2} L^{-1} \left[\frac{1}{S+2} \right]$$

$$= \frac{1}{2} = e^{-t} + \frac{1}{2} e^{-2t}$$

Exercises :-

1) $\frac{1}{(S-3)^5}$

2) $\frac{S}{(S-b)^2 + a^2}$

3) $\frac{(S+d)}{(S+a)^2 + b^2}$

10.3 Solving ordinary differential Equations using Laplace Transform.

Laplace transformation can be used to solve ordinary differential equations with constant coefficients :-

1) The method is illustrated by means of Examples :-

1) Solve the equation $\frac{d^2y}{dt^2} + \frac{2dy}{dt} - 3y = \sin t$.

given that $y \frac{dy}{dt} = 0$ when $t = 0$.

The equation can be written in the form $y'' + 2y' - 3y = \sin t$

Applying Laplace transforms to both sides. We have

$$L(y'' + 2y' - 3y) = L(\sin t)$$

$$(i.e.) L(y'') + 2L(y') - 3L(y) = \frac{1}{S^2 + 1}$$

$$S^2 L(y) - S y(0) - y'(0) + 2S \{S L(y) - y(0)\}$$

$$- 3L(y) = \frac{1}{S^2 + 1}$$

Substituting the values of $y(0)$ and $y'(0)$ in the equations

$$S^2 \bar{y} + 2S \bar{y} - 3 \bar{y} = \frac{1}{S^2 + 1} \text{ where } \bar{y} = L(y)$$

$$(S^2 + 2S - 3) \bar{y} = \frac{1}{S^2 + 1}$$

$$\bar{y} = \frac{1}{(S^2 + 2S - 3)(S^2 + 1)} = \frac{1}{(S + 3)(S - 1)(S^2 + 1)}$$

$$\therefore y = L^{-1} \frac{1}{(S - 1)(S + 3)(S^2 + 1)}$$

On splitting into partial fractions we get

$$y = L^{-1} \left(\frac{-1}{40} \frac{1}{S+3} + \frac{1}{8} \frac{1}{S-1} + \frac{1}{10} \frac{S}{S^2+1} - \frac{1}{5} \frac{1}{S^2+1} \right)$$

$$= -\frac{1}{40} L^{-1} \left(\frac{1}{S+3} \right) + \frac{1}{8} L^{-1} \left(\frac{1}{S-1} \right) - \frac{1}{10} L^{-1} \left(\frac{S}{S^2+1} \right) - \frac{1}{5} L^{-1} \left(\frac{1}{S^2+1} \right)$$

$$= -\frac{1}{40} e^{-3t} + \frac{1}{8} e^t - \frac{1}{10} \cos t - \frac{1}{5} \sin t$$

Laplace transform can also be used to solve simultaneous linear differential equations.

Example :

1. Solve the simultaneous equations

$$3 \frac{dx}{dt} + \frac{dy}{dt} + 2x = 1 \quad \dots\dots\dots (1)$$

$$\frac{dx}{dt} + 4 \frac{dy}{dt} + 3y = 0 \quad \dots\dots\dots (2)$$

given $x = 0 = y$ at $t = 0$.

Applying Laplace transforms to both the equations (1) becomes

$$3 L(x') + L(y') + 2 L(x) = L(1).$$

$$3 [S\bar{x} - x(0)] + s\bar{y} - y(0) + 2\bar{x} = 1/s \text{ where } \bar{x} = L(x)$$

Since $x(0) = 0$; $y(0) = 0$ we have

$$3s\bar{x} + S\bar{y} + 2\bar{x} = \frac{1}{S}$$

$$(\text{i.e.}) (3S + 2)\bar{x} + S\bar{y} = \frac{1}{S} \quad \dots\dots\dots (3)$$

Equation (2) becomes.

$$L(x') + 4 L(y') + 3L(y) = 0.$$

$$S\bar{x} - x(0) + 4 \{S\bar{y} - y(0)\} + 3\bar{y} = 0.$$

$$S\bar{x} + 4s\bar{y} + 3\bar{y} = 0$$

$$S\bar{x} + (4S+3)\bar{y} = 0 \quad \dots\dots\dots (4)$$

Solving (3) and (4) we get

$$\bar{x} = \frac{4S+3}{S(S+1)(11S+6)} \quad \text{and} \quad \bar{y} = -\frac{1}{(11S+6)(S+1)}$$

$$\therefore x = L^{-1} \left\{ \frac{4S+3}{S(S+1)(11S+6)} \right\}$$

$$= L^{-1} \left\{ \frac{1}{2} \cdot \frac{1}{S} - \frac{1}{5} \cdot \frac{1}{S+1} - \frac{33}{10} \cdot \frac{1}{11S+6} \right\}$$

$$= \frac{1}{2} L^{-1} \left(\frac{1}{S} \right) - \frac{1}{5} L^{-1} \left(\frac{1}{S+1} \right) - \frac{33}{10} L^{-1} \left(\frac{1}{11S+6} \right)$$

$$= \frac{1}{2} - \frac{1}{5} e^{-t} - \frac{3}{10} e^{-\frac{6t}{11}}$$

$$y = L^{-1} \left\{ \frac{-1}{(11S+6)(S+1)} \right\}$$

$$= L^{-1} \left\{ \frac{1}{5} \cdot \frac{1}{s+1} - \frac{11}{5} \cdot \frac{1}{11s+6} \right\}$$

$$= \frac{1}{5} L^{-1} \left(\frac{1}{s+1} \right) - \frac{11}{5} \cdot \frac{1}{11} L^{-1} \left(\frac{1}{S+6/11} \right)$$

$$= \frac{1}{5} e^{-t} - \frac{1}{5} e^{-\frac{6}{11}t}$$

Laplace transform can be used to solve differential equations with variable coefficients

we have show that

$$L \{t f(t)\} = - \frac{d}{ds} L \{f(t)\}$$

$$\text{and } L \{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} L \{f(t)\}$$

These results care used to solve equations containing variable coefficients.

The following worked out examples will illustrate the method.

Example (1) :

Solve the equation

$$t \frac{d^2 y}{dt^2} - (2+t) \frac{dy}{dt} + 3y = (t-1) \text{ when } y(0)=0$$

Taking Laplace transforms on both sides.

We have

$$L(ty'') - L(2+t)y' + 3L(y) = L(t-1)$$

$$\text{(ie) } - \frac{d}{ds} \{S^2 L(y)\} - sy(0) - y'(0) - 2 \{SL(y) - y(0)\}$$

$$+ \frac{d}{ds} \{(SL(y) - y(0))\} + 3L(y) = \frac{1}{S^2} - \frac{1}{S}$$

$$\text{Let } L(y) = \bar{y}$$

Putting $y(0)=0$ we have

$$- \frac{d}{ds} \{S^2 \bar{y} - y'(0)\} - 2(s\bar{y}) + \frac{d}{ds} (s\bar{y}) + 3\bar{y} = \frac{1-S}{s^2}$$

$$\text{(ie) } -s^2 \frac{d\bar{y}}{ds} - 2s\bar{y} - 2s\bar{y} + s \frac{d\bar{y}}{ds} + \bar{y} + 3\bar{y} = \frac{1-s}{s^2}$$

$$\text{(ie) } - (s^2-s) \frac{d\bar{y}}{ds} - 4(s-1)\bar{y} = - \frac{(s-1)}{s^2}$$

$$\text{(ie) } \frac{d\bar{y}}{ds} + \frac{4\bar{y}}{s} = \frac{1}{s^3}$$

Solving this equation $\bar{y} = \frac{1}{2} \cdot \left(\frac{1}{s^2} \right) + \left(\frac{C}{s^4} \right)$

$$\therefore y = \frac{1}{2} L^{-1} \left(\frac{1}{s^2} \right) + C L^{-1} \left(\frac{1}{s^4} \right)$$

$$= \frac{t}{2} + \frac{ct^3}{6}$$

Hence $y = \frac{t}{2} + At^3$ where A is an arbitrary constant

Certain equations involving integrals can also be solved by Laplace transform:

Determine y which satisfies the equation

$$\frac{dy}{dt} + 3y + 2 \int_0^t y dt = t \text{ for which } y(0) = 0$$

Taking Laplace transforms on both sides we get

$$L(y') + 3L(y) + 2L \int_0^t y dt = L(t)$$

$$sL(y) - y(0) + 3L(y) + \frac{2}{s} L(y) = \frac{1}{s^2}$$

Putting $L(y) = \bar{y}$ and substituting $y(0) = 0$ we have

$$s\bar{y} + 3\bar{y} + \frac{2\bar{y}}{s} = \frac{1}{s^2}$$

$$\bar{y} \left(s + 3 + \frac{2}{s} \right) = \frac{1}{s^2}$$

$$\bar{y} = \frac{1}{s(s+1)(s+2)}$$

$$y = L^{-1} \frac{1}{s(s+1)(s+2)}$$

$$= L^{-1} \left(\frac{1}{2} \cdot \frac{1}{s} - \frac{1}{s+1} + \frac{1}{2} \cdot \frac{1}{s+2} \right)$$

$$= \frac{1}{2} L^{-1} \left(\frac{1}{s} \right) - L^{-1} \left(\frac{1}{s+1} \right) + \frac{1}{2} L^{-1} \left(\frac{1}{s+2} \right)$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

Exercises :

Solve the following differential equations :

1) $\frac{d^2y}{dt^2} + 4 \frac{dy}{dt} - 5y = 5$ given that $y=0, \frac{dy}{dt} = 2$ when $t=0$

2) $\frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + 5y = 4e^{-t}$ given that $y = \frac{dy}{dt} = 0$ when $t=0$

3) $\frac{d^2y}{dx^2} - 10 \frac{dy}{dx} + 24y = 24x$ given that $y = \frac{dy}{dx} = 0$ when $x=0$

4) $\frac{d^3y}{dt^3} + \frac{dy}{dt} = e^{-2t}$ subject to the conditions $y(0) = y'(0) = y''(0) = 0$

5) $\frac{d^3y}{dt^3} - 3 \frac{dy}{dt} + 2y = 4e^{2t}$ with the conditions $y(0) = -3y'(0) = 5$

Books for Reference :-

Differential equation

S. Narayan

T.K. Manicavachagam Pillay.